Supplemental Material for: "Spectrum estimation of density operators with alkaline-earth atoms"

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S1. MEAN-FIELD THEORY

Here we derive Eq. (2) in the main text, when elastic $e-g$ and $e-e$ interactions are included in the dark Hamiltonian H_D. We remind the reader that the mean-field analysis is not valid for small n. Without loss of generality, let's work in the nuclear spin basis, where the initial nuclear-spin density matrix is diagonal: $\rho^{mm'} = \delta_{mm'}p_m$. After the first pulse, we then have

$$\rho_{gg}^{mm'} = \delta_{mm'}p_m\cos^2(\beta/2), \quad \text{(S1)}$$
$$\rho_{ee}^{mm'} = \delta_{mm'}p_m\sin^2(\beta/2), \quad \text{(S2)}$$
$$\rho_{ge}^{mm'} = \delta_{mm'}p_m\frac{i}{2}\sin\beta. \quad \text{(S3)}$$

The generalized dark Hamiltonian is [1]

$$\begin{aligned}\hat{H}_D &= U_{gg}\sum_{j<k}\hat{\sigma}_{gg}^j\hat{\sigma}_{gg}^k(1-s_{jk}) + U_{ee}\sum_{j<k}\hat{\sigma}_{ee}^j\hat{\sigma}_{ee}^k(1-s_{jk}) - \delta\sum_k\hat{\sigma}_{ee}^k \\ &+Vn_gn_e + V\sum_{pq,j\neq k}\hat{c}_{jep}^\dagger\hat{c}_{kep}\hat{c}_{kgq}^\dagger\hat{c}_{jgq} + V^{ex}\sum_{pq,jk}\hat{c}_{jgp}^\dagger\hat{c}_{keq}^\dagger\hat{c}_{jgq}\hat{c}_{kep} + V^{ex}\sum_{pq,j\neq k}\hat{c}_{kgp}^\dagger\hat{c}_{jeq}^\dagger\hat{c}_{jgq}\hat{c}_{kep}, \quad \text{(S4)}\end{aligned}$$

where j and k are sites and p and q are nuclear spins. The constants are given by $U_{gg} = 4\pi\hbar a_{gg}\omega_\perp/L$, $U_{ee} = 4\pi\hbar a_{ee}\omega_\perp/L$, $V = 4\pi\hbar\frac{(a_{eg}^+ + a_{eg}^-)}{2}\omega_\perp/L$, and $V^{ex} = 4\pi\hbar\frac{(a_{eg}^+ - a_{eg}^-)}{2}\omega_\perp/L$. Here a_{eg}^+ is the s-wave scattering length between atoms in a symmetric electronic (g, e) configuration, a_{eg}^- is the s-wave scattering length between atoms in an anti-symmetric electronic (g, e) configuration and a_{ee} is the s-wave scattering length between e atoms. Note that U_{gg} is written as U in the main text for brevity. The evolution equations during the dark time are

$$\begin{aligned}\dot{\rho}_{\alpha\beta}^{mm'} &= i\delta(\delta_{\alpha e}-\delta_{\beta e})\rho_{\alpha\beta}^{mm'} \\ &-iU_{gg}(n-1)\left[\sum_r\delta_{\alpha g}(\rho_{\alpha\beta}^{mm'}\rho_{gg}^{rr} - \rho_{\alpha\beta}^{rm'}\rho_{gg}^{mr}) - \sum_r\delta_{\beta g}(\rho_{\alpha\beta}^{mm'}\rho_{gg}^{rr} - \delta_{\alpha\beta}^{mr}\rho_{gg}^{rm'})\right] \\ &-iU_{ee}(n-1)\left[\sum_r\delta_{\alpha e}(\rho_{\alpha\beta}^{mm'}\rho_{ee}^{rr} - \rho_{\alpha\beta}^{rm'}\rho_{ee}^{mr}) - \sum_r\delta_{\beta e}(\rho_{\alpha\beta}^{mm'}\rho_{ee}^{rr} - \delta_{\alpha\beta}^{mr}\rho_{ee}^{rm'})\right] \\ &-iV(n-1)\left[(\delta_{\alpha g}-\delta_{\beta g})\sum_r\rho_{ee}^{rr} + (\delta_{\alpha e}-\delta_{\beta e})\sum_r\rho_{gg}^{rr}\right]\rho_{\alpha\beta}^{mm'} \\ &iV(n-1)\sum_r(\delta_{\alpha e}\rho_{g\beta}^{rm'}\rho_{eg}^{mr} - \delta_{\beta g}\rho_{\alpha e}^{mr}\rho_{eg}^{rm'} + \delta_{\alpha g}\rho_{e\beta}^{rm'}\rho_{ge}^{mr} - \delta_{\beta e}\rho_{\alpha g}^{mr}\rho_{ge}^{rm'}) \\ &iV^{ex}(n-1)\sum_r(\delta_{\alpha g}\rho_{g\beta}^{rm'}\rho_{ee}^{mr} - \delta_{\beta g}\rho_{\alpha g}^{mr}\rho_{ee}^{rm'} + \delta_{\alpha e}\rho_{e\beta}^{rm'}\rho_{gg}^{mr} - \delta_{\beta e}\rho_{\alpha e}^{mr}\rho_{gg}^{rm'}) \\ &-iV^{ex}(n-1)\sum_r(\delta_{\alpha e}\rho_{g\beta}^{mm'}\rho_{eg}^{rr} - \delta_{\beta g}\rho_{\alpha e}^{mm'}\rho_{eg}^{rr} + \delta_{\alpha g}\rho_{e\beta}^{mm'}\rho_{ge}^{rr} - \delta_{\beta e}\rho_{\alpha g}^{mm'}\rho_{ge}^{rr}). \quad \text{(S5)}\end{aligned}$$

Since there are no $m \neq m'$ components in the beginning of the dark time [see Eqs. (S1-S3)], we see that these components also stay zero during the dark time. The remaining evolution equations are

$$\begin{aligned}
\dot{\rho}_{gg}^{mm} &= iV^{ex}(n-1)(\rho_{ge}^{mm}\sum_r \rho_{eg}^{rr} - \rho_{eg}^{mm}\sum_r \rho_{ge}^{rr}), && \text{(S6)}\\
\dot{\rho}_{ee}^{mm} &= iV^{ex}(n-1)(\rho_{eg}^{mm}\sum_r \rho_{ge}^{rr} - \rho_{ge}^{mm}\sum_r \rho_{eg}^{rr}) && \text{(S7)}\\
&= -\dot{\rho}_{gg}^{mm}, && \text{(S8)}\\
\dot{\rho}_{ge}^{mm} &= -i\delta\rho_{ge}^{mm} - iU_{gg}(n-1)\rho_{ge}^{mm}\sum_{r\neq m}\rho_{gg}^{rr} + iU_{ee}(n-1)\rho_{ge}^{mm}\sum_{r\neq m}\rho_{ee}^{rr}\\
&\quad -iV(n-1)\rho_{ge}^{mm}\sum_{r\neq m}(\rho_{ee}^{rr}-\rho_{gg}^{rr}) - iV^{ex}(n-1)(\rho_{ee}^{mm}-\rho_{gg}^{mm})\sum_{r\neq m}\rho_{ge}^{rr}, && \text{(S9)}
\end{aligned}$$

In terms of the matrix elements at the end of the dark time τ, the measurement result (after the last pulse of area $-\beta$) is

$$\begin{aligned}
\frac{\langle \hat{n}_e\rangle}{n} &= \frac{1}{2}\left(1 + \sum_m(\rho_{ee}^{mm}(\tau) - \rho_{gg}^{mm}(\tau))\cos\beta - i\sum_m(\rho_{eg}^{mm}(\tau) - \rho_{ge}^{mm}(\tau))\sin\beta\right) && \text{(S10)}\\
&\to \frac{1}{2}\left(1 - \cos^2\beta - i\sum_m(\rho_{eg}^{mm}(\tau) - \rho_{ge}^{mm}(\tau))\sin\beta\right), && \text{(S11)}
\end{aligned}$$

where the last line holds only for $\Gamma_{ee} = 0$, in which case the total number of g atoms and total number of e atoms are both conserved during the dark time (in e-e losses, the total number of e atoms is not conserved).

S2. EYD SPECTRUM ESTIMATION

Here we calculate $\Pr(\vec{\lambda}|n,\vec{p})$ exactly for finite n for EYD measurement. This is required for Fig. 2(b) and Fig. 2(c) in the main text, along with the general calculation of $\langle \hat{n}_e(\tau,\vec{p})\rangle/n$ via Eq. (12), used to generate Fig. 2(a).

To carry out the analysis, first note that the measurement projectors $\{\Pi_{\vec{\lambda}}\}$ commute with the action of spin-rotation $\hat{V}^{\otimes n}$ applied to all spins. Therefore the measurement outcome is independent of the eigenstates of $\hat{\rho}$, and for the purpose of calculation we can take it to be $\hat{\rho} = \sum_{i=1}^N p_i|i\rangle\langle i|$. Thus the overall state of the system is

$$\begin{aligned}
\hat{\rho}^{\otimes n} &= \left(\sum_{i=1}^N p_i|i\rangle\langle i|\right)^{\otimes n} && \text{(S12)}\\
&= \sum_{m_1,m_2,\ldots,m_N|n} p_1^{m_1}p_2^{m_2}\ldots p_N^{m_N}\ \hat{M}_{m_1,m_2,\ldots,m_N}, && \text{(S13)}
\end{aligned}$$

where the sum is over all non-negative integers $\{m_1,m_2,\ldots,m_N\}$ such that $\sum_{i=1}^N m_i = n$, and $\hat{M}_{m_1,m_2,\ldots,m_N}$ is the projector onto the subspace of states containing m_i spin-state i's (i.e. the state $|1\rangle^{\otimes m_1}|2\rangle^{\otimes m_2}\ldots|N\rangle^{\otimes m_N}$, and all distinct permutations). Note that the subspace $\mathcal{H}_{m_1,m_2,\ldots,m_N} \subset \mathcal{H}$ which $\hat{M}_{m_1,m_2,\ldots,m_N}$ projects onto is preserved by the action of any permutation $\sigma \in S_n$, and therefore supports a representation of S_n. As such, $\mathcal{H}_{m_1,m_2,\ldots,m_N}$ can be decomposed into irreps of S_n. For $c(\vec{\lambda}|\vec{m})$ copies of the $\vec{\lambda}$ irrep of S_n in $\mathcal{H}_{m_1,m_2,\ldots,m_N}$, the probability of obtaining measurement outcome $\vec{\lambda}$ is:

$$\Pr(\vec{\lambda}|n,\vec{p}) = \mathrm{Tr}\left(\Pi_{\vec{\lambda}}\hat{\rho}^{\otimes n}\right) = \|\vec{\lambda}_{S_n}\|\sum_{m_1,m_2,\ldots,m_N|n} p_1^{m_1}p_2^{m_2}\ldots p_N^{m_N}\ c(\vec{\lambda}|\vec{m}), \tag{S14}$$

where we remind the reader that $\Pi_{\vec{\lambda}}$ is the projector onto the subspace $\mathcal{H}_{\vec{\lambda}} \subset \mathcal{H}$, which carries the $\vec{\lambda}$-irrep of $S_n \times SU(N)$. Defining $l_i := \lambda_i + N - i$, the dimension of the $\vec{\lambda}$ irrep of S_n is

$$\|\vec{\lambda}_{S_n}\| = \frac{n!}{l_1!\cdots l_N!}\prod_{i<j}^N (l_i - l_j), \tag{S15}$$

which can be calculated directly for a particular instance $\vec{\lambda}$.

To obtain $c(\vec{\lambda}|\vec{m})$, first note that $c(\vec{\lambda}|\vec{m})$ cannot depend on the ordering of the integers in $\vec{m} = (m_1, m_2, ..., m_N)$. Therefore it is sufficient to consider $c(\vec{\lambda}|\vec{\mu})$ for which $\mu_i \geq \mu_{i+1}$, therefore $\vec{\mu}$ specifies a valid Young diagram. Consider filling the n boxes of the Young diagram $\vec{\lambda}$ with integers. We call the resulting filled Young diagram a *semi-standard Young tableau* if and only if the numbers are non-decreasing across rows from left to right, and strictly increasing down columns. Then $c(\vec{\lambda}|\vec{\mu})$ is the *Kostka number* $K_{\lambda\mu}$ [2], which is given by the number of distinct semi-standard Young tableaux that can be constructed by filling Young diagram $\vec{\lambda}$ with μ_1 1's, μ_2 2's etc. This can be calculated numerically for particular instances of $\vec{\lambda}$ and $\vec{\mu}$.

In the special case of $N = 2$, taking $(\lambda_1, \lambda_2) = (\frac{n}{2} + S, \frac{n}{2} - S)$, the expression for $\Pr(\vec{\lambda}|n, \vec{p}) = \Pr(S|n, \vec{p})$ takes a simple form. In this case, $\|\vec{\lambda}_{S_n}\| = \binom{n}{\lambda_1} - \binom{n}{\lambda_1+1}$ and $c(\vec{\lambda}|\vec{\mu})$ is zero for $\mu_1 > \lambda_1$, and unity for $\mu_1 \leq \lambda_1$. Therefore,

$$\Pr(S|n, \vec{p}) \;=\; \left[\binom{n}{\frac{n}{2}+S} - \binom{n}{\frac{n}{2}+S+1}\right] \sum_{m=\frac{n}{2}-S}^{\frac{n}{2}+S} p^m (1-p)^{n-m}. \tag{S16}$$

This is used to generate Fig. 2(b) in the main text.

S3. EVALUATING Tr $(\rho^{\otimes n}\mathcal{B}_w)$

In the main text, we require the evaluation of $\mathrm{Tr}\left(\rho^{\otimes n}\mathcal{B}_w\right)$ where $\mathcal{B}_w := e^{i\alpha \sum_{j=1}^{n-1-w}(1-s_{jn})}$ in order to calculate $\langle \hat{n}_e(\tau, \vec{p})\rangle/n$ in Eq. (10). We provide two approaches to analyze $\mathrm{Tr}\left(\rho^{\otimes n}\mathcal{B}_w\right)$. In the main text we cover the first approach, which uses group representation theory, and in Sec. S3 A we provide a technical step required for the proof which was omitted from the main text. In Sec. S3 B we give an alternative analysis of $\mathrm{Tr}\left(\rho^{\otimes n}\mathcal{B}_w\right)$ which we use to prove that the deviation of Eq. (2) in the main text from the exact result is $\tilde{\mathcal{O}}(1/\sqrt{n})$.

A. Using group representation theory

In the main text in the paragraphs following Eq. (12) we prove that

$$\mathrm{Tr}_\lambda\left[\mathcal{B}_w\right] = \sum_\xi m(\lambda, \xi) \sum_{r=1}^{N} \frac{\|\xi^{-r}\|}{\|\lambda\|} v_r(\xi), \tag{S17}$$

where $v_r(\xi)$ is the eigenvalue of $\mathcal{B}_w$ on the irrep ξ^{-r}. Here we prove the claim in the main text that $v_r(\xi) = e^{i\alpha(n-w-1)}e^{-i\alpha(\xi_r - r)}$. In order to compute $v_r(\xi)$, it is necessary to understand the irrep ξ^{-r} of S_{l-1} inside the irrep ξ of S_l, where $l := n - w$. To this end, we construct a series of spaces of tabloids. Recall that given a Young diagram $\xi = (\xi_1, ..., \xi_N)$ with $\sum_r \xi_r = l$, a *Young tableau* t is formed by inserting integers in the boxes of ξ. Here we consider those Young tableaux with each number from 1 to l appearing in precisely one box of ξ. A *tabloid* $\{t\}$ is an equivalence class of Young tableaux t, where two tableaux are equivalent if one is obtained from another by permuting within each row. In other words, if A_t is the *group of all row-preserving permutations* of t, then $\{t\} = \{\alpha t : \alpha \in A_t\}$. The symmetric group S_l acts on the set of all tabloids by permuting numbers; it can be verified that $\{\pi t\} = \{\pi\alpha t\}$ for any $\alpha \in A_t$ and $\pi \in S_l$, and hence the notation $\pi\{t\}$ makes sense. Let B_t be the *group of all column-preserving permutations* of t, and define

$$e_t = \sum_{\beta \in B_t} \mathrm{sgn}(\beta)\beta\{t\},$$

which is called a *polytabloid*. The action of S_l on the span of all polytabloids is isomorphic to the irrep ξ. A basis for this irrep can be chosen to be $\{e_t : t \text{ is a standard Young tableau}\}$. (A *standard tableau* is one in which numbers are increasing in each row and column.)

Define V_i to be the span of e_t where t is a standard Young tableau with n in one of the rows $1, \ldots, i$. Certainly, $V_1 \subseteq V_2 \subseteq \cdots V_N = \xi$. Observe that V_i is a representation space of S_{l-1} because the position of the number l is fixed by S_{l-1}. It is known that V_i/V_{i-1} is isomorphic to ξ^{-i} [3]. Define $h := \sum_{j=1}^{l-1} s_{jl}$. Note that h preserves each V_i, because V_i and its orthogonal complement contain distinct irreps of S_{l-1}, and the projection Π_{V_i} onto V_i from ξ can be written by some element of $\mathbb{C}S_{l-1}$, which implies that h commutes with the projector Π_{V_i}.

The eigenvalue v_r is determined by $he_t = u_r e_t + w$, with $v_r = \exp(-i\alpha u_r)$, for some $e_t \in V_r \setminus V_{r-1}$ and $w \in V_{r-1}$. We will read off the coefficient of $\{t\}$, where 'l' is placed in the row i of a standard tableau t. (If it is not possible for such t to be standard, then $V_r/V_{r-1} = 0$.) Since

$$he_t = \sum_{\tau\in h,\ \beta\in B_t} \mathrm{sgn}(\sigma)\tau\beta\{t\}, \tag{S18}$$

we see that the coefficient of $\{t\}$ in he_t is

$$u_r = \sum_{\tau\in h,\ \beta\in B_t\ :\ \tau\beta\{t\}=\{t\}} \mathrm{sgn}(\beta) = \sum_{\tau\in h,\ \beta\in B_t\ :\ \tau\beta\in A_t} \mathrm{sgn}(\beta). \tag{S19}$$

In order to make a nonzero contribution to the sum, τ must be a member of $B_t \cdot A_t$. If both $\alpha \in B_t$ and $\beta \in A_t$ are nontrivial, then $\beta\alpha$ cannot be a transposition. Thus, $\tau = \beta\alpha$ must be a member of either A_t, in which case $\mathrm{sgn}(\beta = 1) = 1$, or B_t, in which case $\mathrm{sgn}(\beta) = \mathrm{sgn}(\tau) = -1$. There are $\xi_r - 1$ terms of h that belong to A_t, and $r-1$ terms of h that belong to B_t. Therefore,

$$u_r = (\xi_r - 1)(+1) + (r-1)(-1) = \xi_r - r. \tag{S20}$$

As $\mathcal{B}_w = e^{i\alpha\sum_{j=1}^{n-1-w}(1-s_{jn})} = e^{i\alpha(n-w-1)}e^{-i\alpha h}$, we see that $v_r(\xi) = e^{i\alpha(n-w-1)}e^{-i\alpha(\xi_r - r)}$ as required.

B. Using elementary analysis

Our goal here is to show that the large-n form of $\langle n_e(\tau,\vec{p})\rangle/n$ is that of the mean field result Eq. (2) in the main text, with a deviation which decreases as $\tilde{\mathcal{O}}(1/\sqrt{n})$. First we calculate the expectation value of a permutation operator $P(\sigma)$, defined as

$$P(\sigma) = \sum_{y_1=1}^{d}\cdots\sum_{y_n=1}^{d} |y_{\sigma^{-1}(1)}, y_{\sigma^{-1}(2)}, \dots, y_{\sigma^{-1}(m)}\rangle\langle y_1, y_2, \dots, y_m|, \tag{S21}$$

for permutation σ. Let $\sigma = \sigma_1\sigma_2\cdots\sigma_m$ be the decomposition into disjoint cycles. Some σ_j may be 1-cycle. By $|\sigma_j|$ we denote the length of a cycle. For example, we have $|(1)| = 1$, $|(56)| = 2$ and $|(245)| = 3$. The following equation is simple and useful,

$$\mathrm{Tr}(P(\sigma)\rho^{\otimes m}) = \prod_j \mathrm{Tr}(\rho^{|\sigma_j|}). \tag{S22}$$

This is particularly simple to evaluate, since $\mathrm{Tr}(\rho^l) = \sum_{r=1}^{d} p_r^l$, for $(p_1, p_2, \dots, p_d)$ the spectrum of ρ. To prove this, it suffices to verify that (i) $P(\sigma) = P(\sigma_1)P(\sigma_2)\cdots P(\sigma_n)$ where distinct $P(\sigma_j)$ are supported on disjoint tensor factors, and (ii) if $\sigma = \sigma_1$ is a cycle of length m, then $\mathrm{Tr}(P(\sigma)\rho^{\otimes m}) = \mathrm{Tr}(\rho^m)$. The truth of (i) is evident. For (ii), we may assume $\sigma = (123\cdots m)$. Then,

$$\begin{aligned}\mathrm{Tr}(P(\sigma)\rho^{\otimes m}) &= \sum_{\{y_j\}} \langle y_1, y_2, \dots, y_m|\rho^{\otimes m}|y_m, y_1, \dots, y_{m-1}\rangle \\ &= \sum_{\{y_j\}} \rho_{y_1y_m}\rho_{y_2y_1}\cdots\rho_{y_my_{m-1}} = \mathrm{Tr}(\rho^m).\end{aligned}$$

Next, we proceed to evaluate $\mathrm{Tr}\left(\rho^{\otimes n}\mathcal{B}_w\right)$ where $\mathcal{B}_w := e^{i\alpha\sum_{j=1}^{n-1-w}(1-s_{jn})} = e^{i\alpha(n-w-1)}e^{-i\alpha\sum_{j=1}^{n-1-w}s_{jn}}$ by expanding the exponential. Let $m = n-w$ and $z = -i(m-1)\alpha$. Hereafter in this section, we denote by $\langle\cdot\rangle := \mathrm{Tr}(\rho^{\otimes m}\ \cdot\ )$ the expectation value with respect to $\rho^{\otimes m}$,

$$\langle\mathcal{B}_w\rangle = e^{i\alpha(m-1)}\langle e^{(z/(m-1))\sum_{j=1}^{m-1}s_{j,m}}\rangle = \frac{e^{i\alpha(m-1)}}{m}\sum_{k=1}^{m}\langle e^{(z/(m-1))\sum_{j\neq k}^{m}s_{j,k}}\rangle \tag{S23}$$

$$= e^{i\alpha(m-1)}\sum_{l=0}^{\infty}\frac{z^l}{l!}\underbrace{\left\langle\left(\frac{1}{m(m-1)^l}\sum_{j_1,j_2,\dots,j_l\neq k}^{m} s_{j_1,k}s_{j_2,k}\cdots s_{j_l,k}\right)\right\rangle}_{X_l}. \tag{S24}$$

The operator X_l contains precisely $m(m-1)^l$ terms in the sum. Each summand is some permutation operator $\sigma \in S_m$, and $\langle X_l \rangle$ can be interpreted as the average value $\langle \sigma \rangle$ upon a random choice of σ among $m(m-1)^l$ possibilities. (This probability distribution has nothing to do with $\Pr(w|n,\beta)$ above.) From Eq. (S22), we know that $\langle \sigma \rangle$ depends only on the lengths of cycles in the disjoint cycle decomposition of σ. If $j_1, \ldots, j_l$ are all distinct, then $\sigma = (j_1 k) \cdots (j_l k) = (k j_l j_{l-1} \cdots j_1)$ is a cycle of length $l+1$, and $\langle \sigma \rangle = \operatorname{Tr} \rho^{l+1}$. If m is sufficiently large, then this is the most typical case. Indeed, the probability that the $j_1, \ldots, j_l$ are all distinct (i.e. the probability that one obtains σ of length $l+1$) is

$$p(m,l) = \frac{(l+1)!\binom{m}{l+1}}{m(m-1)^l} = \frac{(m-1)(m-2)\cdots(m-l)}{(m-1)^l} \geq 1 - \frac{l^2}{m-1}.$$

This allows us to bound the "error"

$$\Delta_l := |\langle X_l \rangle - \operatorname{Tr} \rho^{l+1}| \leq (1 - p(m,l)) \cdot \max_{\sigma : |\sigma| \leq l} |\langle \sigma \rangle - \operatorname{Tr}(\rho^{l+1})| \leq \frac{2l^2}{m-1},$$

where we used the trivial normalization $\operatorname{Tr}(\rho^{l+1}) \leq 1$ and $\langle \sigma \rangle \leq 1$. Therefore,

$$\begin{aligned} \langle \mathcal{B}_w \rangle = e^{i\alpha(m-1)} \sum_{l=0}^{\infty} \frac{z^l}{l!} \langle X_l \rangle &= e^{i\alpha(m-1)} \left[\sum_{l=0}^{\infty} \frac{z^l}{l!} \operatorname{Tr}(\rho^{l+1}) + \frac{2}{m-1} \mathcal{O}\left(\sum_{l=0}^{\infty} \frac{|z|^l l^2}{l!} \right) \right] \\ &= e^{i\alpha(m-1)} \left[\sum_{r=1}^{d} p_r e^{z p_r} + \mathcal{O}\left(\frac{\exp(|z|)}{m} \right) \right]. \end{aligned} \tag{S25}$$

This proves that in the limit when $m = n - w$ is large, for fixed $m\alpha$,

$$\mathfrak{Re}\,\{e^{i\delta\tau} \operatorname{Tr}(\rho^{\otimes n} \mathcal{B}_w)\} \;\to\; \underbrace{\sum_r p_r \cos[\alpha(m-1)(1-p_r) + \delta\tau]}_{\mathcal{C}_w} + \mathcal{O}(m^{-1}), \tag{S26}$$

where we also have defined $\mathcal{C}_w$. Recall that from Eq. (10) in the main text, we must sum over w according to the binomial distribution $\Pr(w|n,\beta)$ in order to obtain $\langle \hat{n}_e(\tau, \vec{p}) \rangle / n$. We then see that

$$|\mathcal{C}_w - \mathcal{C}_{w'}| \leq |(n-1)\alpha| \cdot \frac{|w - w'|}{n-1}, \tag{S27}$$

which is implied by the Taylor series (mean-value theorem) with respect to w.

Using the tail bound for binomial distribution

$$\sum_{w : |w - \bar{w}| > (n-1)\epsilon} \Pr(w|n,\beta) \leq 2e^{-2(n-1)\epsilon^2},$$

we arrive at the proof of the convergence of $\langle n_e \rangle / n$ for large n:

$$\left| \frac{\langle \hat{n}_e \rangle}{n} - \frac{\sin^2 \beta}{2} \left[1 - \sum_{r=1}^{d} p_r \cos\left(\cos^2 \frac{\beta}{2} \alpha(n-1)(1-p_r) + \delta\tau \right) \right] \right| \tag{S28}$$

$$\leq \frac{1}{2} \sum_w \Pr(w|n,\beta) |\mathfrak{Re}\,\{e^{i\delta\tau} \operatorname{Tr}(\rho^{\otimes n} \mathcal{B}_w)\} - \mathcal{C}_{\bar{w}-1}| \tag{S29}$$

$$\leq 2e^{-2(n-1)\epsilon^2} + \frac{1}{2} \max_{w : |w - \bar{w}| \leq (n-1)\epsilon} |\mathfrak{Re}\,\{e^{i\delta\tau} \operatorname{Tr}(\rho^{\otimes n} \mathcal{B}_w)\} - \mathcal{C}_w| + |\mathcal{C}_w - \mathcal{C}_{\bar{w}-1}| \tag{S30}$$

$$\leq 2e^{-2(n-1)\epsilon^2} + \frac{\mathcal{O}(\exp(n\alpha))}{n} + 2(n\alpha)\epsilon \qquad \text{for any } \epsilon > 0 \tag{S31}$$

$$\leq \mathcal{O}\left(\frac{\exp(n\alpha)}{\sqrt{n / \log n}} \right) \qquad \text{setting } \epsilon^2 = \frac{\log n}{n}. \tag{S32}$$

Therefore we have shown that Eq. (2) in the main text differs from the exact result by $\tilde{\mathcal{O}}(1/\sqrt{n})$ in the limit $n \to \infty$ while holding $n\alpha$ constant. Recall that the tilde above the $\mathcal{O}$ means we neglect logarithmic factors. There are a few comments on the technical aspects of the analysis above. If β is sufficiently small such that $n \sin^2 \frac{\beta}{2}$ is a constant irrespective of n, then $1/\sqrt{n}$ scaling is improved to be $1/n$. This is because the binomial distribution has smaller relative deviation when the probability is small.

S4. NUMERICAL CALCULATION OF $\langle n_e(\tau, \vec{p})\rangle/n$

Here we collect the equations necessary to calculate $\langle n_e(\tau, \vec{p})\rangle/n$ for the convenience of the reader. This is used in the main text to generate plots, for example Fig. 2(a). We also show how to evaluate $\langle n_e(\tau, \vec{p})\rangle/n$ more efficiently for large n approximately by taking advantage of the fact that it is calculated in terms of narrow distributions.

By substituting Eq. (11) into Eq. (10) in the main text,

$$\frac{\langle \hat{n}_e(\tau, \vec{p})\rangle}{n} = \frac{\sin^2\beta}{2}\left[1 - \sum_{w=0}^{n-1} \Pr(w|n,\beta)\, \mathfrak{Re}\left\{e^{i\delta\tau} \sum_{\lambda} \Pr(\vec{\lambda}|n,\vec{p})\ \mathrm{Tr}_\lambda(\mathcal{B}_w)\right\}\right], \tag{S33}$$

where $\Pr(w|n,\beta) := \binom{n-1}{w}\cos^{2(n-w-1)}\frac{\beta}{2}\sin^{2w}\frac{\beta}{2}$. In Eq. (S14) in Sec. S2 we showed that

$$\Pr(\vec{\lambda}|n,\vec{p}) = \|\vec{\lambda}_{S_n}\| \sum_{m_1,m_2,\ldots,m_d|n} p_1^{m_1} p_2^{m_2} \ldots p_d^{m_d}\ c(\vec{\lambda}|\vec{m}), \tag{S34}$$

where $c(\vec{\lambda}|\vec{\mu})$ is the Kostka number, given by the number of distinct semi-standard Young tableaux that can be constructed by filling Young diagram $\vec{\lambda}$ with μ_1 1's, μ_2 2's etc, and [repeating Eq. (S15)] the irrep dimension is

$$\|\vec{\lambda}_{S_n}\| = \frac{n!}{l_1! \cdots l_N!}\prod_{i<j}^{N}(l_i - l_j), \qquad \text{with } l_i := \lambda_i + N - i. \tag{S35}$$

The final step is to substitute for $\mathrm{Tr}_\lambda(\mathcal{B}_w)$ as in Eq. (11) in the main text

$$\mathrm{Tr}_\lambda(\mathcal{B}_w) = e^{i\alpha(n-w-1)} \sum_{\xi} \frac{m(\lambda,\xi)\|\xi\|}{\|\lambda\|} \sum_{r=1}^{d} \frac{\|\xi^{-r}\|}{\|\xi\|} e^{-i\alpha(\xi_r - r)}, \tag{S36}$$

where the sum is over all irreps ξ of S_{n-w} and ξ^{-r} is the irrep of S_{n-w-1} defined by removing a box from the r-th row of irrep ξ of S_{n-w}. The multiplicity $m(\lambda,\xi)$ is calculated iteratively from the branching rules which state that the restriction of an irrep λ of S_l to S_{l-1} consists of distinct irreps λ^{-r} of S_{l-1} with multiplicity 1. Therefore, $m(\lambda,\xi)$ is the number of distinct paths from λ to ξ, where each step in a path is a Young diagram, with one box removed from the previous step.

In the main text we show how to calculate $\langle \hat{n}_e\rangle/n$ exactly, here we describe how to drop terms to improve the efficiency of the calculation without sacrificing much accuracy. We assume that d is held fixed, and that n becomes large here. In the main text, we introduced three probability distributions $\Pr(\lambda|n,p)$, $\Pr(w|n,\beta)$, and $\Pr(\xi|w,\lambda)$, all of which turn out to be unimodal for large n. The first one $\Pr(\lambda|n,p)$ is concentrated at $\lambda \simeq n\vec{p}$ with the deviation of $\|\vec{\lambda}/n - \vec{p}\|$ being $\mathcal{O}(n^{-\frac{1}{2}})$ by the result of EYD algorithm [4, 5]. By retaining only terms within a few standard deviations of $\vec{p}$ the number of $\vec{\lambda}$ that need to be summed over drops from $\sim \frac{1}{nd}\left(\frac{e^2 n}{d^2}\right)^d \sim n^{d-1}$ [6] to approximately $\mathcal{O}(n^{(d-1)/2})$. The second distribution $\Pr(w|n,\beta)$ is the familiar binomial distribution. By including only terms within a few standard deviations of the mean, $\bar{w} = (n-1)\sin^2\frac{\beta}{2}$, we reduce the number of w which are summed from $\sim n$ to $\mathcal{O}(n^{-\frac{1}{2}})$.The third distribution $\Pr(\xi|w,\lambda)$ is concentrated at $\xi \simeq \frac{n-w}{n}\lambda$ with the deviation $\|\frac{\vec{\xi}}{n-w} - \frac{\vec{\lambda}}{n}\|$ being $\mathcal{O}(n^{-\frac{1}{2}})$. There are $\mathcal{O}(n^{(d-1)/2})$ terms within a few standard deviations of the mean, as opposed to (what we expect to be) the full $\sim n^{d-1}$ terms. Together therefore, the total number of terms after excluding those which contribute negligibly is reduced from $\sim n^{2d-1}$ to $\mathcal{O}(n^{(2d-1)/2})$.

S5. EFFECTS OF IMPERFECTIONS

In this section we describe the effects of two main types of imperfections on the proposal, namely deviation from an exact square-well potential, and particle loss. We will rely on numerics to analyze these cases as many of the symmetries which rendered our analysis tractable do not apply. For simplicity we consider there to be only two nuclear spin degrees of freedom, i.e., $d = 2$.

First consider the case of a non-square well potential without loss. The Hamiltonian in Eq. (1) of the main text is replaced by

$$\hat{H}_{\mathrm{D}} = \sum_{j<k} U_{jk}\hat{\sigma}_{gg}^{j}\hat{\sigma}_{gg}^{k}(1 - \hat{s}_{jk}) - \delta\sum_{k}\hat{\sigma}_{ee}^{k}, \tag{S37}$$

where the strength of interaction has picked up mode dependence because the modes no longer are precise sinusoidal functions. In Fig. S1(a) we plot $\langle \hat{n}_e(t)\rangle/n$ for each of the $n(n-1)/2$ constants U_{jk} chosen uniformly from the interval $[U-dU/2, U+dU/2]$ for a variety of $\frac{dU}{U}$ ratios, where $\langle \hat{n}_e(t)\rangle$ is averaged over realizations. For the experimental parameters in the main text, i.e. with $L\sim 10\mu$m, and using lasers with wavelength close to 600 nm, we can estimate two extremal values of $L_\pm \approx (10\pm 0.6)\mu$m. From the relation $U=(4\pi a_{gg}\omega_\perp)/L$, we can thereby estimate $dU \approx (4\pi a_{gg}\omega_\perp)/L_- - (4\pi a_{gg}\omega_\perp)/L_+ \approx 0.12U$. From Fig. S1(a) it is clear that the deviation in $n_e(t)$ due to dU depends strongly on the time t. To estimate how much the typical $dU/U = 0.12$ impacts the estimation of p, we therefore fix the time $t=1/U$ Fig. S1(b) shows the average $\langle \hat{n}_e(1/U)\rangle/n$, plus and minus its standard deviation (over realizations of U_{jk} chosen uniformly from the interval $[U-dU/2, U+dU/2]$ for $dU/U=0.12$). The largest deviations in the estimated p occur near $p=1/2$, where an uncertainty of ± 0.05 results from $dU/U=0.12$.

Now consider the case of particle loss (but with $U_{jk}=U$ for all j,k for simplicity). We write the evolution of the n-atom density matrix ρ as,

$$\dot{\rho} = -i[\hat{H}_D,\rho] - \frac{\Gamma}{2}\sum_{i<j}\left(\hat{c}^\dagger_{ij}\hat{c}_{ij}\rho + \rho\hat{c}^\dagger_{ij}\hat{c}_{ij} - 2\hat{c}_{ij}\rho\hat{c}^\dagger_{ij}\right), \tag{S38}$$

where Γ is the loss rate under lossy e-e collisions [7], and where c_{ij} is written in terms of atomic annihilation operators,

$$\hat{c}_{ij} = \frac{1}{\sqrt{2}}\left(\hat{c}_{ie\downarrow}\hat{c}_{je\uparrow} - \hat{c}_{ie\uparrow}\hat{c}_{je\downarrow}\right). \tag{S39}$$

For small n, one can calculate this evolution exactly; see Fig. S1(c). In the (experimentally relevant) parameter regime of $\Gamma/U=0.5$, there is significant deviation compared with the loss-free case.

To study loss for large n, we consider a mean-field approximation to Eq. (S38). We remind the reader that the mean-field analysis is not valid for small n. A part of this approximation is to assume the density matrix is separable,

$$\rho = \otimes_{l=1}^n\left[\rho(l)_{in,in}|in\rangle\langle in| + \rho(l)_{in,out}|in\rangle\langle out| + \rho(l)_{out,in}|out\rangle\langle in| + \rho(l)_{out,out}|out\rangle\langle out|\right], \tag{S40}$$

where we have introduced another degree of freedom $\{|in\rangle, |out\rangle\}$ to track whether a particle is in the trap or has been lost, and $\rho(l)_{\alpha\beta}$ is a density matrix for a single atom l with electronic and nuclear degrees of freedom.

Now consider taking the trace over all but the jth particle in the right hand side of Eq. (S38). The terms $\langle in|\mathrm{Tr}_{n\backslash l}(-i[\hat{H}_D,\rho])|in\rangle$, $\langle in|\mathrm{Tr}_{n\backslash l}(\hat{c}^\dagger_{ij}\hat{c}_{ij}\rho)|in\rangle$ and $\langle in|\mathrm{Tr}_{n\backslash l}(\rho\hat{c}^\dagger_{ij}\hat{c}_{ij})|in\rangle$ have contributions only from density matrices $\rho(l)_{in,in}$ since $\hat{H}_D$ implicitly includes a projection onto atoms in the trap. Here, $\mathrm{Tr}_{n\backslash l}(\cdot)$ implies tracing out the degrees of freedom on all atoms, except for atom l. On the other hand, the term $\langle in|\mathrm{Tr}_{n\backslash l}(\hat{c}_{ij}\rho\hat{c}^\dagger_{ij})|in\rangle$ must be zero since the recycling term outputs states in $|out\rangle$, which are cancelled by the projection $\langle in|\cdot|in\rangle$. Therefore Eq. (S38) becomes

$$\dot{\rho}(l)_{in,in} = \langle in|\mathrm{Tr}_{n\backslash l}(-i\hat{H}'\rho + i\rho\hat{H}'^\dagger)|in\rangle,\ \text{where } \hat{H}' = -\delta\sum_k\hat{\sigma}^k_{ee} + \sum_{j<k}\left[U\hat{\sigma}^j_{gg}\hat{\sigma}^k_{gg}(1-\hat{s}_{jk}) - \frac{i\Gamma}{4}\hat{\sigma}^j_{ee}\hat{\sigma}^k_{ee}(1-\hat{s}_{jk})\right]. \tag{S41}$$

From here on, we drop the in subscript on single-particle density operators. Then,

$$\begin{aligned}\dot{\rho}(l) &= i\delta\left[\hat{\sigma}^l_{ee}\rho(l) - \rho(l)\hat{\sigma}^l_{ee}\right] + \sum_{j<k(j\neq l,k\neq l)}\frac{-\Gamma}{2}A_e(j,k)\rho(l) + \\ &\quad + \sum_{j=1(j\neq l)}^n -iU\left[B_g(l,k) - C_g(l,k)\right] - \frac{\Gamma}{4}\left[B_e(l,k)+C_e(l,k)\right].\end{aligned} \tag{S42}$$

where U and Γ are defined to be real, and where we define,

$$A_\gamma(j,k) = \mathrm{Tr}_{jk}\left[\hat{\sigma}^j_{\gamma\gamma}\hat{\sigma}^k_{\gamma\gamma}(1-\hat{s}_{jk})\rho(j)\otimes\rho(k)\right] = \mathrm{Tr}_{jk}\left[\rho(j)\otimes\rho(k)\hat{\sigma}^j_{\gamma\gamma}\hat{\sigma}^k_{\gamma\gamma}(1-\hat{s}_{jk})\right], \tag{S43}$$

$$B_\gamma(l,k) = \mathrm{Tr}_k\left[\hat{\sigma}^l_{\gamma\gamma}\hat{\sigma}^k_{\gamma\gamma}(1-\hat{s}_{lk})\rho(l)\otimes\rho(k)\right] = \mathrm{Tr}_k\left[\hat{\sigma}^k_{\gamma\gamma}\hat{\sigma}^l_{\gamma\gamma}(1-\hat{s}_{kl})\rho(k)\otimes\rho(l)\right], \tag{S44}$$

$$C_\gamma(l,k) = \mathrm{Tr}_k\left[\rho(l)\otimes\rho(k)\hat{\sigma}^l_{\gamma\gamma}\hat{\sigma}^k_{\gamma\gamma}(1-\hat{s}_{lk})\right] = \mathrm{Tr}_k\left[\rho(k)\otimes\rho(l)\hat{\sigma}^k_{\gamma\gamma}\hat{\sigma}^l_{\gamma\gamma}(1-\hat{s}_{lk})\right]. \tag{S45}$$

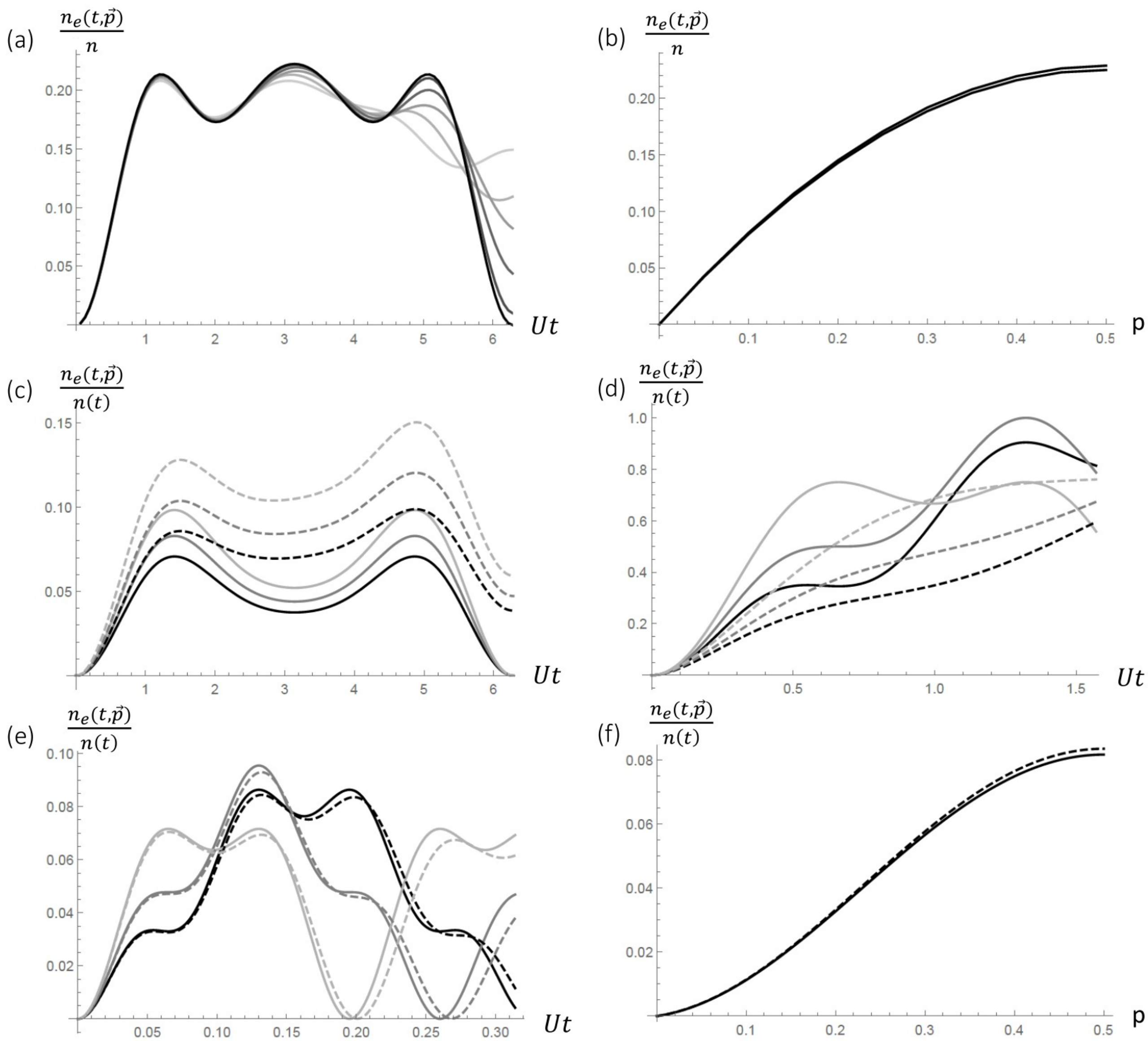

FIG. S1: Plots of $n_e(t)/n(t) = \langle \hat{n}_e(t,\vec{p})\rangle / \langle \hat{n}(t)\rangle$. In all plots, $\delta = 0$. (a) For $n = 4$, and a variety of values of $dU/U = 0, 0.1, 0.2, 0.3, 0.4, 0.5$ (dark to light), using $\vec{p} = (2/3, 1/3)$ and $\beta = \pi/4$. (b) To estimate the error that results from $dU/U = 0.12$, we plot the mean $n_e(t)$, plus and minus the standard deviation over realizations at fixed time $t = 1/U$ for $n = 4$ as a function of p in $\vec{p} = (1-p, p)$, with $\beta = \pi/4$ as in (a). The largest uncertainties in estimating p are expected to occur for p near $1/2$ as the change in n_e due to non-zero dU is largest, and also the sensitivity of n_e with respect to p is least in that region. (c) For $n = 4$, with $\Gamma/U = 0$ (solid) and $\Gamma/U = 0.5$ (dashed) using a variety of values of $\vec{p} = (4/5, 1/5), (3/4, 1/4), (2/3, 1/3)$ (dark to light). Here, $\delta = 0$ and $\beta = \pi/4$. (d) For $n = 20$, with $\Gamma/U = 0$ (solid) and $\Gamma/U = 0.5$ (dashed) for a variety of values of $\vec{p} = (4/5, 1/5), (3/4, 1/4), (2/3, 1/3)$ (dark to light). Here, $\delta = 0$ and $\beta = \pi/4$. The shape is altered significantly by Γ. (e) As in (d), but with $n = 100$ and $\beta = \pi/20$. The effect of Γ is much less pronounced. (f) To estimate the error that results from $\Gamma/U = 0.5$, we plot the mean $n_e(t)/n(t)$ for fixed time $t = 0.05/U$ for $n = 100$ as a function of p in $\vec{p} = (1-p, p)$, and compare with the case for $\Gamma = 0$ (dashed). Here, $\beta = \pi/4$ as in (e) Those spectra with p close to $1/2$ are most sensitive to loss.

Each of these can be calculated explicitly,

$$A_\gamma(j,k) = \sum_{mn} \rho^{mm}_{\gamma\gamma}(j)\rho^{nn}_{\gamma\gamma}(k) - \rho^{nm}_{\gamma\gamma}(j)\rho^{mn}_{\gamma\gamma}(k), \tag{S46}$$

$$[B_\gamma(l,k)]^{pp'}_{\eta\eta'} = \delta_{\eta\gamma} \sum_n \rho^{pp'}_{\gamma\eta'}(l)\rho^{nn}_{\gamma\gamma}(k) - \rho^{np'}_{\gamma\eta'}(l)\rho^{pn}_{\gamma\gamma}(k), \tag{S47}$$

$$[C_\gamma(l,k)]^{pp'}_{\eta\eta'} = \delta_{\eta'\gamma} \sum_n \rho^{pp'}_{\eta\gamma}(l)\rho^{nn}_{\gamma\gamma}(k) - \rho^{pn}_{\eta\gamma}(l)\rho^{np'}_{\gamma\gamma}(k). \tag{S48}$$

Finally, we use these to find the mean-field equations of motion in the case in which $\rho(l)$ is independent of l,

$$\begin{aligned} \frac{d}{dt}\rho^{pp'}_{\eta\eta'} &= i\delta\left(\delta_{\eta e}\rho^{pp'}_{e\eta'} - \delta_{\eta' e}\rho^{pp'}_{\eta e}\right) + \frac{(n-1)(n-2)}{2}\left(\frac{-\Gamma}{2}\right)\rho^{pp'}_{\eta\eta'}\sum_{mn}\left(\rho^{mm}_{ee}\rho^{nn}_{ee} - \rho^{nm}_{ee}\rho^{mn}_{ee}\right) + \\ &- iU(n-1)\left[\delta_{\eta g}\sum_n\left(\rho^{pp'}_{g\eta'}\rho^{nn}_{gg} - \rho^{np'}_{g\eta'}\rho^{pn}_{gg}\right) - \delta_{\eta' g}\sum_n\left(\rho^{pp'}_{\eta g}\rho^{nn}_{gg} - \rho^{pn}_{\eta g}\rho^{np'}_{gg}\right)\right] + \\ &- \frac{\Gamma}{4}(n-1)\left[\delta_{\eta e}\sum_n\left(\rho^{pp'}_{e\eta'}\rho^{nn}_{ee} - \rho^{np'}_{e\eta'}\rho^{pn}_{ee}\right) + \delta_{\eta' e}\sum_n\left(\rho^{pp'}_{\eta e}\rho^{nn}_{ee} - \rho^{pn}_{\eta e}\rho^{np'}_{ee}\right)\right]. \end{aligned} \tag{S49}$$

We use this to make the plots in Fig. S1(d), and observe that even for moderate $n = 20$, non-zero $\Gamma/U = 0.5$ alters the observed outcomes significantly. Three possible approaches to overcome this problem are: (1) As described in the main text, reduce the radial trap strength for the excited atoms during the dark time. (2) Use a small β, which should help since the collisional effects arise at $\mathcal{O}(\beta^4)$, whereas the signal scales as $\sim \beta^2$. This has the downside of requiring more data to be taken to accommodate the reduced signal; Fig. S1(e). (3) Account for the modified evolution introduced by finite Γ by including it in the model and using fits to the modified model to extract the spectrum. To estimate the uncertainty introduced in the estimation of p by loss $\Gamma/U = 0.5$ in the case in which a small tipping angle $\beta = \pi/20$ is used, we plot $\langle \hat{n}_e(t)\rangle/\langle \hat{n}(t)\rangle$ as a function of p for fixed time $t = 0.05/U$, and compare it with what would be expected if there was no loss in Fig. S1(f). The largest deviations in the estimated p occur near $p = 1/2$, where a systematic shift of -0.05 results from $\Gamma/U = 0.5$, however one could account for the corrections introduced by the known non-zero Γ.

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