Supplemental material

In this supplement, we present technical details omitted from the main text. In section S.I, we derive the qubit-photon Hamiltonian within the low-energy continuum approximation. In section S.II, we discuss the ultraviolet cutoff. In section S.III, we derive Eq. (8) for the curvature-limited correlations in the limit $L=1$. In section S.IV, we derive the bound-state equations for a single qubit and for two qubits. In section S.V, we derive the spontaneous emission rate of an excited qubit within the continuum approximation—Eq. (9) from the main text.

S.I. HAMILTONIAN WITHIN THE CONTINUUM APPROXIMATION

In this section, we derive the qubit-photon Hamiltonian within the low-energy continuum approximation developed in Ref. [S1].

We embed the hyperbolic lattice into the Poincare disk $\mathbb{D}=\{z\in\mathbb{C},|z|<1\}$. Sums over lattice sites can be approximated by integrals in the continuum using the following relation

$$\sum_i \to \frac{28}{\pi}\int \frac{d^2z}{(1-|z|^2)^2}. \tag{S1}$$

The discrete bosonic operator a_i (satisfying $\left[a,a^\dagger\right]=1$) can then be approximated by a continuum field operator

$$a(z_i)=\sqrt{\frac{28}{\pi}}a_i, \tag{S2}$$

which satisfies the commutation relation

$$\left[a(z),a^\dagger(z')\right]=(1-|z|^2)^2\delta^2(z-z'). \tag{S3}$$

As shown in Ref. [S1], the tight-binding Hamiltonian from Eq. (1) of the main text is well-approximated by

$$\hat{H}_{\text{ph}}=-t\sum_{<ij>\in\mathcal{G}}a_i^\dagger a_j \to \int \frac{d^2z}{(1-|z|^2)^2}\; a^\dagger(z)\left(-3t-t\frac{3}{4}h^2\Delta_g\right)a(z), \tag{S4}$$

where $\Delta_g=(1-|z|^2)^2(\partial_x^2+\partial_y^2)$ is the hyperbolic Laplacian.

The full Hamiltonian from Eq. (1) is then given by

$$\hat{H}=\Delta\sum_{i\in\mathcal{S}}|{\uparrow}\rangle\langle{\uparrow}|_i+\tilde{g}\sum_{j\in\mathcal{S}}\left(\sigma_j^+a(z_j)+\text{h.c.}\right)+\hat{H}_{\text{ph}}, \tag{S5}$$

$$\hat{H}_{\text{ph}}=\int \frac{d^2z}{(1-|z|^2)^2}\; a^\dagger(z)\left(-3t-t\frac{3}{4}h^2\Delta_g\right)a(z), \tag{S6}$$

where $\tilde{g}=g\sqrt{\frac{\pi}{28}}$.

We will also need the Hamiltonian in Fourier space, valid for the infinite system ($L=1$). In that case, we have the following Fourier transform relations [S2]:

$$a(z)=\int\frac{d^2k}{(2\pi)^2}\tanh\left(\frac{\pi k}{2}\right)\psi_K(z)a(K), \tag{S7}$$

$$a(K)=\int d^2z\frac{1}{(1-|z|^2)^2}\psi_K^*(z)a(z). \tag{S8}$$

Here, $K=ke^{\mathrm{i}\beta}$ or $\mathbf{k}=k(\cos\beta,\sin\beta)^T$, and

$$\psi_K(z)=\psi_{\mathbf{k}}(z)=\left(\frac{1-|z|^2}{|1-ze^{-\mathrm{i}\beta}|^2}\right)^{\frac{1}{2}(1+\mathrm{i}k)} \tag{S9}$$

is an eigenfunction of the hyperbolic Laplacian with eigenvalue $-(1+k^2)$. The Fourier transformed operators satisfy

$$\left[a(K),a^\dagger(K')\right]=\frac{(2\pi)^2}{\tanh\left(\frac{\pi k}{2}\right)}\delta^2(K-K'). \tag{S10}$$

The Hamiltonian then becomes

$$\hat{H} = \Delta \sum_{i\in\mathcal{S}} |{\uparrow}\rangle\langle{\uparrow}|_i + \tilde{g}\sum_{j\in\mathcal{S}} \int \frac{d^2k}{(2\pi)^2} \tanh\Big(\frac{\pi k}{2}\Big)\Big(\psi_K(z_j)a(z_j)\sigma_j^+ + \text{h.c.}\Big) + \hat{H}_{\text{ph}}, \tag{S11}$$

$$\hat{H}_{\text{ph}} = \int \frac{d^2k}{(2\pi)^2} \tanh\Big(\frac{\pi k}{2}\Big)\omega(k)a^\dagger(K)a(K), \tag{S12}$$

where

$$\omega(k) = -3t + t\frac{3}{4}h^2(k^2+1). \tag{S13}$$

S.II. ULTRAVIOLET CUTOFF

In this section, we discuss the ultraviolet (UV) cutoff.

Let us first fix the UV cutoff Λ in

$$G(z,z',\lambda) = -\int_{k\leq\Lambda} \frac{\mathrm{d}^2k}{(2\pi)^2} \tanh\Big(\frac{\pi k}{2}\Big) \frac{\psi_K(z)\psi_K^*(z')}{\lambda-(k^2+1)}. \tag{S14}$$

We insert $z = z' = z_1$ and find

$$G(z_1,z_1,\lambda) = -\int_{k\leq\Lambda} \frac{\mathrm{d}^2k}{(2\pi)^2} \frac{\tanh(\frac{\pi k}{2})}{\lambda-(k^2+1)}. \tag{S15}$$

Compare this to the continuum approximation result [S1]

$$G_{ij}(\omega) = M\frac{\pi}{28}G\Big(z_i,z_j,\lambda = M(\omega+3)\Big), \tag{S16}$$

where $M = \frac{4}{3h^2} = 17.529$ for the $\{7,3\}$ lattice. We conclude that the lattice Green function for $L \to 1$ can be approximated by

$$G_{11}(\omega) = -M\frac{\pi}{28}\int_{k\leq\Lambda} \frac{\mathrm{d}^2k}{(2\pi)^2} \frac{\tanh(\frac{\pi k}{2})}{M(\omega+3)-(k^2+1)}. \tag{S17}$$

To fix Λ, we insert $\omega = -3$, which is below the band edge and so leads to a finite integral. We define

$$C(\ell) = G_{11}(-3) \overset{!}{=} M\frac{\pi}{28}\int_{k\leq\Lambda} \frac{\mathrm{d}^2k}{(2\pi)^2} \frac{\tanh(\frac{\pi k}{2})}{k^2+1}. \tag{S18}$$

As a function of the number ℓ of rings we have:

ℓ	1	2	3	4	5	6
$C = G_{11}(-3)$	0.448	0.649	0.707	0.728	0.736	0.739
Λ	4.59	8.87	10.7	11.4	11.7	11.9
Λ_1	4.06	7.89	9.52	10.2	10.5	10.6

We evaluate the integral and find

$$C = \frac{M}{56}\int_0^\Lambda \mathrm{d}k\ k\ \frac{\tanh(\frac{\pi k}{2})}{k^2+1}, \tag{S19}$$

which determines Λ. If we neglect the tanh term, then

$$C \approx \frac{M}{56}\int_0^{\Lambda_1} \mathrm{d}k\ k\ \frac{1}{k^2+1} = \frac{M}{112}\log\big(1+\Lambda_1^2\big), \tag{S20}$$

which defines

$$\Lambda_1 = \sqrt{e^{112C/M}-1} \simeq e^{56C/M}. \tag{S21}$$

Hence the generic value for $\Lambda \simeq \Lambda_1$ is 10, and we have $\sqrt{M} = 4.2$, and so

$$\Lambda \gtrsim 2\sqrt{M}. \tag{S22}$$

S.III. CURVATURE-LIMITED CORRELATIONS

In this section, we derive Eq. (8) for the curvature-limited correlations in the limit $L = 1$. As shown in Ref. [S1], the correlation function $G(z, z,' \omega)$ for $L = 1$ only depends on the hyperbolic distance $d(z, z')$. It is given by

$$G(z, z', \omega) = \frac{1}{2\pi}\mathrm{Re}\Big[Q_\nu\Big(\cosh(d(z,z')/\kappa)\Big) - \mathcal{C}(\omega)\ P_\nu\Big(\cosh(d(z,z')/\kappa)\Big)\Big],$$

where P_ν (Q_ν) are Legendre functions of the first (second) kind, $\nu = \frac{1}{2}(-1 + \mathrm{i}\sqrt{\mathrm{M}(\omega + 3 - \frac{1}{\mathrm{M}})})$, and $\mathcal{C}(\omega)$ is a constant. We use the fact that the lower band edge is located at $E_0 = -3 + \frac{1}{M} + \mathcal{O}(h^3)$ for $L = 1$ and write $\nu = \frac{1}{2}(-1 + \mathrm{i}\sqrt{\mathrm{M}(\omega - \mathrm{E}_0)})$. We neglect the $\mathcal{O}(h^3)$ corrections in the following.

Let us first consider the special case $\omega = E_0$ and show that the correlation length remains finite. We have $\nu = -1/2$ and find that $\mathcal{C}(E_0) = -\mathrm{i}\pi/2$ is purely imaginary while $P_{-1/2}(y)$ is purely real. Hence the term $\mathcal{C}P_\nu$ does not contribute to the Green function for $\omega = E_0$. We have Re $Q_{-1/2}(x) \sim \pi/\sqrt{2x}$ for $x \to \infty$, or

$$\mathrm{Re}\ Q_{-1/2}\Big(\cosh(d/\kappa)\Big) \sim \pi e^{-d/(2\kappa)} \tag{S23}$$

as $d/\kappa \to \infty$. Consequently, for $\omega = E_0$, the correlation length satisfies $\xi(E_0) = \kappa$, which is finite.

Now consider frequencies $\omega < E_0$ below the band edge. We observe that $\nu = -\frac{1}{2}(1 + \sqrt{M(E_0 - \omega)})$ is real and negative. The function Re $Q_\nu(x)$ scales like x^ν for $x \to \infty$. Hence

$$\mathrm{Re}\ Q_\nu\Big(\cosh(d/\kappa)\Big) \approx \cosh(d/\kappa)^\nu \approx e^{d\nu/\kappa}, \tag{S24}$$

or

$$\frac{1}{\xi(\omega)} \approx \frac{1}{\kappa}\Big(1 + \sqrt{M(E_0 - \omega)}\Big). \tag{S25}$$

We independently verified this scaling behavior numerically from fitting the long-distance decay of $G(r, 0, \omega)$ for $\omega \lesssim E_0$.

S.IV. BOUND STATES

In this section, we derive the bound-state equations for a single qubit and for two qubits—Eqs. (6) and (12) in the main text.

A. Bound state for a single qubit

Working directly in the continuum and in Fourier space (valid for $L = 1$), we can write down the single-excitation wavefunction

$$|\psi\rangle = \left[\int_K u(K)a^\dagger(K) + c_1\sigma_1^+\right]|{\downarrow}\, 0\rangle\,, \tag{S26}$$

where $\int_K \equiv \int \frac{d^2k}{(2\pi)^2}\tanh(\frac{\pi k}{2})$. Schroedinger's equation ($\hat{H}\,|\psi_B\rangle = E_B\,|\psi_B\rangle$) yields an equation for the bound-state energy:

$$E_B = \Delta + \Sigma(E_B), \tag{S27}$$

$$\Sigma(E_B) = \tilde{g}^2\int_K \frac{1}{E_B - \omega(k)} = -\frac{4\tilde{g}^2}{3th^2}G\Big(z, z; \lambda = \frac{4}{3th^2}(3t + E_B)\Big), \tag{S28}$$

and for the photonic component of the wavefunction

$$u(K) = c_1\tilde{g}\frac{\psi_K^*(z_1)}{E_B - \omega(k)}. \tag{S29}$$

The remaining coefficient c_1 is determined by the normalization condition.

Using the Fourier transform relation in Eq. (S7), the bound-state wavefunction can also be written in real space and is given in Eq. (6) of the main text.

B. Two qubits

For two qubits (at positions 1 and 2, respectively), the single-excitation wavefunction can be written as

$$|\psi\rangle = \left[\int_K u(K)a^\dagger(K) + \sum_{i=1,2} c_i\sigma_i^+\right]|{\downarrow\downarrow}\,0\rangle\,. \tag{S30}$$

Schroedinger's equation can be written as

$$G^{-1}(E_B)\begin{pmatrix}c_1\\c_2\end{pmatrix} = 0, \tag{S31}$$

where the Green function is

$$G(E_B) = \begin{pmatrix}E_B-\omega_q-\Sigma_{11}(E_B) & -\Sigma_{12}(E_B)\\ -\Sigma_{21}(E_B) & E_B-\omega_q-\Sigma_{22}(E_B)\end{pmatrix}^{-1} \tag{S32}$$

and the self energies are

$$\Sigma_{ij}(E_B) = \int_K \frac{\tilde{g}^2\psi_K(z_i)\psi_K^*(z_j)}{E_B-\omega(k)} = -\frac{4\tilde{g}^2}{3th^2}G\Big(z_i, z_j; \lambda = \frac{4}{3th^2}(3t+E_B)\Big). \tag{S33}$$

Eq. (S32) can be written as follows:

$$G(E_B) = \begin{pmatrix}E_B-\omega_q-\Sigma_{11}(E_B) & -\Sigma_{12}(E_B)\\ -\Sigma_{21}(E_B) & E_B-\omega_q-\Sigma_{22}(E_B)\end{pmatrix}^{-1} = \left[(E_B-\Delta-\Sigma_{11}(E_B))\mathbb{1} - \Sigma_{12}(E_B)\sigma_x\right]^{-1}, \tag{S34}$$

where $\mathbb{1}$ is the 2×2 identity matrix and σ_x is the Pauli x matrix. The eigenstates are $\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}1\\-1\end{pmatrix}$. Hence, the bound-state energies $E_B^\pm$ of the symmetric and antisymmetric bound states are given by the poles of

$$\frac{1}{E_B^\pm - \Delta - \Sigma_{11}(E_B^\pm) \mp \Sigma_{12}(E_B^\pm)}. \tag{S35}$$

The wavefunctions are given by

$$|\psi_\pm\rangle = c_\pm\left[\tilde{g}^2\int_K \frac{\psi_K^*(z_1)\pm\psi_K^*(z_2)}{E_B^\pm-\omega(k)}a^\dagger(K) + (\sigma_1^+\pm\sigma_2^+)\right]|{\downarrow\downarrow}\,0\rangle\,, \tag{S36}$$

where $c_\pm$ are normalization constants. The real-space version is given in the main text, in Eq. (12).

S.V. SPONTANEOUS EMISSION AND $j(\omega)$

In this section, we derive the spontaneous emission rate of an excited qubit within the continuum approximation—Eq. (9) from the main text.

We consider a single qubit and work in the continuum approximation in Fourier space (for $L=1$). For weak coupling g, the photons can be integrated out within a Born-Markov approximation, giving rise to a Lindblad master equation for the reduced density matrix of the qubit described by (neglecting the Lamb shift)

$$\partial_t\rho = -i[\Delta\,|{\uparrow}\rangle\langle{\uparrow}|\,,\rho] + \Gamma\sigma^-\rho\sigma^+ - \frac{\Gamma}{2}\{\sigma^+\sigma^-,\rho\}. \tag{S37}$$

The decay rate Γ is given in terms of the spectral function as follows:

$$\Gamma = \frac{j(\Delta)}{2}, \tag{S38}$$

where the spectral function is defined as (this can also be obtained from Fermi's golden rule)

$$j(\omega) = \pi \sum_K |g_K|^2 \delta(\omega - \omega_k). \tag{S39}$$

Here, ω_k describes the dispersion of the photons, given in Eq. (S13), whereas g_K can be read off from Eq. (S11), giving $g_K = g\sqrt{\frac{\pi}{28}}\psi_K(z)$.

Explicitly, we find

$$j(\omega) = \pi g^2 \frac{\pi}{28} \int \frac{d^2k}{(2\pi)^2} \tanh\left(\frac{\pi k}{2}\right) |\psi_K|^2 \delta(\omega - \omega_k) \tag{S40}$$

$$= \pi g^2 \frac{\pi}{28} \int_0^\infty \frac{dk}{2\pi} k \tanh\left(\frac{\pi k}{2}\right) \delta(\omega - E_0 + k^2/M) \tag{S41}$$

$$= \pi g^2 \frac{\pi}{28} \int_0^\infty \frac{dk}{2\pi} k \tanh\left(\frac{\pi k}{2}\right) \frac{\delta(k - \sqrt{(\omega - E_0)M})}{2\sqrt{(\omega - E_0)M}/M} \tag{S42}$$

$$= \frac{\pi}{28} g^2 \tanh\left(\frac{\pi\sqrt{(\omega - E_0)M}}{2}\right) \frac{M}{4}. \tag{S43}$$

[1] Igor Boettcher, Przemyslaw Bienias, Ron Belyansky, Alicia J. Kollár, and Alexey V. Gorshkov, "Quantum simulation of hyperbolic space with circuit quantum electrodynamics: From graphs to geometry," Phys. Rev. A **102**, 32208 (2020).

[2] Sigurdur Helgason, "Non-Euclidean Analysis," in *Non-Euclidean Geometries*, edited by Andras Prekopa and Emil Molnar (Springer, 2006).