Supplemental Material for Fast State Transfer and Entanglement Renormalization Using Long-Range Interactions

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APPLICATION TO DIPOLE-DIPOLE INTERACTIONS

In this Supplement, we show that it is possible to realize a protocol similar to that in the main text by using Rydberg atoms. Rydberg atoms can be made to interact with a dipole-dipole interaction that has distance dependence $1/r^3$. This suggests that, using our protocol, we could produce a cube of side length L in a GHZ state in time proportional to $\log L$. We will demonstrate that a realistic physical interaction can yield this result. Many details on Rydberg atoms and their applications in quantum information can be found in Refs. [1–3], and experimental demonstrations can be found in Refs. [4–7]. Our analysis is focused on Rydberg atoms, but much of it should extend to other dipolar systems, such as polar molecules, with appropriate modification of implementation details [8–11].

We select as qubit states the ground state and a highly excited state of a Rydberg atom under a weak electric field, yielding a purely diagonal atomic interaction [1]. The Hamiltonian of a system of such atoms can be written as:

$$H_{\text{int}} = \sum_{i\neq j} H_{ij} = \sum_{i\neq j} \frac{1 - 3\cos^2\theta_{ij}}{r_{ij}^3} Z_i Z_j \equiv \sum_{i\neq j} V_{ij} Z_i Z_j. \tag{S1}$$

Here, r_{ij} is the distance between atoms i and j, while θ_{ij} is the angle between the electric field and the vector separating the two atoms. We have ignored local terms like Z_i and Z_j, which can be removed by applying local rotations. By applying local rotations, this ZZ Hamiltonian can be used to realize CNOT interactions, regardless of whether the overall sign is positive or negative. This is done by applying local rotations to produce a controlled-phase gate and applying Hadamard operations on the target before and after the evolution to yield a controlled-NOT gate [12]. We assume that, while local control fields may be time-dependent, the two-body interaction in Eq. (S1) is active throughout the entire state transfer process. The individual addressing required to perform these local operations was demonstrated in a 3D optical lattice in Ref. [13]. The roughly 5 μm lattice spacing in that work is also an appropriate spacing for the Rydberg interactions we intend to use in our protocol, as it helps to prevent the dipole-dipole interactions from becoming comparable to the energy level spacing.

To apply the protocol in the main text, qubits must be separated into controls and targets. Such separation can be performed using an echoing procedure: first, qubits evolve under H_{int} and then under $-H_{\text{int}}$ for an equal amount of time. However, halfway through the second evolution, a π-pulse (X gate) is applied to either all target qubits or all control qubits. This has the effect of swapping Z for $-Z$. All interactions between controls and controls, or targets and targets, will remain unchanged, but any control-target interactions will be inverted. Thus, during the $-H_{\text{int}}$ time, control-target interactions experience no net evolution, while any control-control or target-target pair evolution due to $+H_{\text{int}}$ is undone. The $-H_{\text{int}}$ evolution time is equal to the initial entangling $+H_{\text{int}}$ time, so the echoing procedure does not change the scaling with L. Even if the negative interaction is not of the same magnitude as the original, we can still accomplish the echoing by adjusting the timescales, and the scaling with L will still not be changed.

To change the sign of the dipole-dipole interaction, realizing $-H_{\text{int}}$, we can encode the computational states into the fine structure of a Rydberg atom. For specificity, we consider the case of Rb^{87} with a weak applied electric field. Ignoring the hyperfine structure, we encode the state $|0\rangle$ in a superposition of $|L=0, J=1/2, m_J=1/2\rangle$ and $|L=1, J=3/2, m_J=3/2\rangle$ created by applying a microwave dressing field, with most of the amplitude being stored in the latter state. The state $|1\rangle$ is then encoded in $|L=1, J=1/2, m_J=1/2\rangle$. All three states have the same principal quantum number. Details can be found in an analogous scheme for polar molecules presented in entry No. 5 of Table II and Fig 3(d) of Ref. [8]. Note that here we are also dropping local Z terms which can be canceled by a local rotation. We have calculated dipole matrix elements for Rb^{87} across a wide range of principal quantum numbers that confirm this scheme remains viable in the Rydberg setting. We also assume that, in addition to changing the overall sign of the interaction, we are able to place qubits in non-interacting electronic ground states to avoid any unwanted interactions or decay from excited states.

If a volume of control qubits exists, this volume will convert a qubit j into a control after time $\pi/2V_j$, where V_j is the sum over all interaction constants V_{ij} for control qubits i. Suppose that enough qubits have been added that the sum of point-to-point interactions is well-approximated by an integral, which is a good approximation in the relevant asymptotic regime. The total interaction on a qubit j in this case can be written as

$$V_j = \int_C V_{ij} \mathrm{d}C. \tag{S2}$$

Here, C is the volume of control qubits. This quantity has the useful property of scale invariance. If all lengths change by a factor λ, then H_{int} changes by the factor λ^{-3} due to its distance dependence. However, the region of integration expands by λ^3, so the final quantity remains unchanged.

We consider expanding a cube of controls, increasing the side length ℓ by a constant factor λ. After this procedure, we obtain a new cube of side length $\lambda\ell$. Qubits outside of the larger cube have no operations performed on them. Once this expansion has been performed, we expand the cube again. Due to scale invariance, the same operation can be performed in identical time. This means that after n expansion steps, the side length will be $\lambda^n\ell$. Therefore, we can construct a cube of side length L in a time proportional to $\log_\lambda(L/\ell)$ as indicated in the main text. The scaling properties of the integral in Eq. (S2) can be used in cases where $\alpha \neq d$ as well. Equation (S2) implies that the time required to construct a cube of side length L will be:

$$t_{\text{GHZ}} \sim \sum_{i=1}^{log_\lambda(L/\ell)} \lambda^{n(\alpha-d)}. \tag{S3}$$

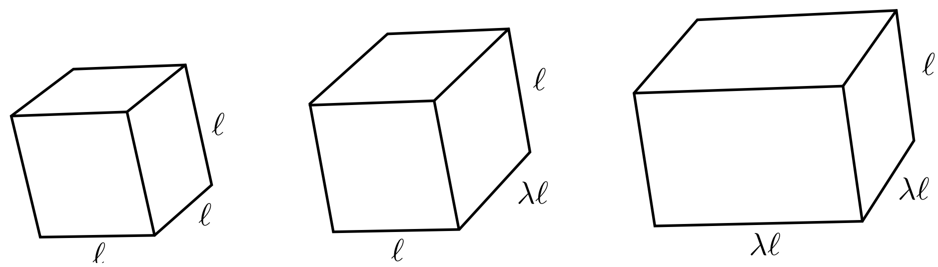

FIG. S1. Successive transformations of the control cube. A cube of side length ℓ is expanded first in one direction, then the next. After the final step (not shown), the result will be a cube of side length $\lambda\ell$.

For $\alpha < d$, this saturates to a number independent of L, and for $\alpha > d$, it implies that $t_{\mathrm{GHZ}} \sim L^{\alpha-d}$. Note that for $\alpha > d+1$, a protocol of successive dilations of the cube fails to provide optimal scaling.

All that remains to be shown is that the size of the cube can be increased by a constant factor in finite time. This is not guaranteed because the dipole-dipole interaction changes sign as a function of θ_{ij}, causing V_j to be zero for qubits at some points. If we could only act with the control cube during the expansion time, we would not be able to perform the expansion as outlined above. However, we can use a slightly more complicated scheme in which some intermediate qubits are used. Rather than expand the entire cube at once, we expand the cube outward in the positive x-, y-, and z-directions successively, each time expanding only to qubits which lie on lines perpendicular to the expanding face of the rectangular prism, as illustrated in Fig. S1. This works because the interaction can be shown to decrease monotonically (in absolute value) along Cartesian directions, as we prove below. Since at long distances we know that the interaction decays to zero and has the same sign for all target qubits, the monotonicity establishes that there is no zero crossing. As there is no zero crossing, there will be a finite time that suffices to complete the expansion. The logarithmic scaling follows.

PROOF OF INTERACTION MONOTONICITY

We will now prove that the interaction between a cube of controls and a target qubit decreases monotonically in Cartesian directions. Suppose we begin with a rectangular prism located in the $y-z$ plane with dimensions $\ell_x \times \ell_y \times \ell_z$ and the origin in the center of one face (see Fig. S2 for an illustration). A qubit at point (x, y, z) then has the interaction integral

$$V = \int_x^{x+\ell_x} \int_{-\ell_y/2+y}^{\ell_y/2+y} \int_{-\ell_z/2+z}^{\ell_z/2+z} \frac{x'^2 + y'^2 - 2z'^2}{(x'^2 + y'^2 + z'^2)^{5/2}} \mathrm{d}x'\mathrm{d}y'\mathrm{d}z'. \tag{S4}$$

The integrand in Eq. (S4) is simply the dipole interaction written in Cartesian coordinates. We choose y and z to fall in $(-\ell_y/2, \ell_y/2)$ and $(-\ell_z/2, \ell_z/2)$ respectively to ensure that their projection to the $y-z$ plane lies on the face of the prism. We consider only positive values of y and z without loss of generality. The derivative of V with respect to

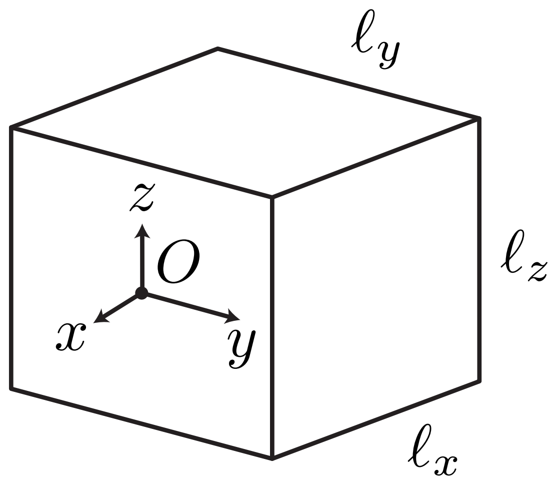

FIG. S2. Illustration of the coordinate system used in this section.

x can be expressed analytically as

$$\partial_x V = D\left(-\frac{\ell_y}{2}+y, -\frac{\ell_z}{2}+z\right) + D\left(\frac{\ell_y}{2}+y, \frac{\ell_z}{2}+z\right) - \left[D\left(-\frac{\ell_y}{2}+y, \frac{\ell_z}{2}+z\right) + D\left(\frac{\ell_y}{2}+y, -\frac{\ell_z}{2}+z\right)\right], \tag{S5}$$

$$D(a,b) = ab\left(\frac{1}{\left((x+\ell_x)^2+c^2\right)\sqrt{(x+\ell_x)^2+a^2+c^2}} - \frac{1}{(x^2+c^2)\sqrt{(x^2+a^2+c^2)}}\right). \tag{S6}$$

For $D(a,b)$, the sign is always determined by the prefactor because the factor in parentheses is strictly negative. Using the fact that y and z must be less than $\ell_y/2$ and $\ell_z/2$ respectively, we can assign a negative sign to the first two D to appear in Eq. (S5) and a positive sign to the second two. Therefore, we find that $\partial_x V$ is always negative in this region, establishing the monotonicity for expansion along one face in the x-direction. This proof also holds for the y-direction immediately from symmetry. For the z-direction, a similar argument holds but with a more complicated parenthetical term in $D(a,b)$.

EFFECTS OF DECOHERENCE

In the next two sections, we will consider the influence of experimental imperfections in qubits and gate operations and examine the implication for our protocols scalability. First, we will consider the influence of decoherence, for instance, due to spontaneous emission out of the Rydberg excited states. The fragile nature of the GHZ state means that a single emission can cause our protocol to fail. We assume that individual qubits fail (spontaneously emit) at a rate γ. This analysis should extend to any similar failure mechanism that occurs at a constant rate. If each expansion step (dilating the cube by λ) takes time δt, then we can consider whether, in the ith timestep, any of the λ^{3i} qubits currently involved emit. If so, we label the step a success. The protocol succeeds if all of its individual steps succeed.

The probability that no spontaneous emissions occur at any of N_t time steps and that the protocol succeeds is

$$P(\text{success}) = \prod_{i=1}^{i=N_t} P(\text{success at step } i) = e^{-\gamma\delta t \sum_i \lambda^{3i}}. \tag{S7}$$

If we demand that the protocol successfully entangle N qubits with a probability $P > \epsilon$, then Eq. (S7) becomes

$$\sum_{i=1}^{\log_\lambda N^{1/3}} \lambda^{3i} = \frac{\lambda^3 (N-1)}{\lambda^3 - 1} < \frac{\ln \frac{1}{\epsilon}}{\gamma \delta t}. \tag{S8}$$

This suggests a limit on the number of qubits which can be entangled with a system of decohering qubits, which we write as

$$N_{\text{lr}} < 1 + \frac{\ln \frac{1}{\epsilon}}{\gamma\delta t} \frac{\lambda^3 - 1}{\lambda^3}. \tag{S9}$$

Here N_{lr} refers to the number of qubits that can be entangled using our long-range interacting protocol. Note that if ϵ and λ are taken to be of order 1, Eq. (S9) simply implies that $N_{\text{lr}}\gamma\delta_t \lesssim 1$, which is unsurprising since our largest entangled state decays in a time $1/N_{\text{lr}}\gamma$. We can also consider what this limit looks like in the case of a protocol which uses nearest-neighbor interactions and, at each step, increases the cube's side length by one. In this case, the ith timestep has i^3 qubits entangled, and there are $N^{1/3}$ such steps. A similar argument to the above leads us to calculate

$$\sum_{i}^{N^{1/3}} i^3 = \frac{1}{4}\left[N^{4/3} + 2N + N^{2/3}\right] < \frac{\ln \frac{1}{\epsilon}}{\gamma\delta t}. \tag{S10}$$

If we assume we're interested in cases where N is somewhat large *a priori*, then we write the following loose bound by dropping strictly positive terms:

$$N_{\text{nn}} < \left(\frac{4 \ln \frac{1}{\epsilon}}{\gamma\delta t}\right)^{3/4}. \tag{S11}$$

Here the exponent $3/4$ arises because we summed over $N^{1/3}$ terms like i^3, yielding $N^{4/3}$ and then inverted that. Suppose we take $\lambda = 2$, in which case the first step of each protocol is the same and we can equate the two δt. Then the ratio of the two threshholds is

$$\frac{N_{\text{lr}}}{N_{\text{nn}}} = \frac{7}{16\sqrt{2}} \left(\frac{\ln \frac{1}{\epsilon}}{\gamma\delta t}\right)^{1/4}. \tag{S12}$$

To evaluate this figure of merit, we can look at the original proposal for interaction-based Rydberg gates, which suggests a two-qubit gate timescale of less than a nanosecond [2]. Our protocol also requires several one qubit gates in each step, which can also be accomplished on nanosecond timescales using pulsed lasers [14]. Estimating $\delta t \sim 5$ ns, demanding a success probability of 1/2, and taking the Rb^{87} 100s state lifetime of 340 μs at a temperature of 300K [1], we find that $N_{\text{lr}}/N_{\text{nn}} \approx 4.5$, meaning that a long-range protocol can achieve a maximally entangled state containing nearly 4.5 times as many qubits as one constructed by nearest-neighbor interactions. This figure rises to 4.9 if we solve Eq. (S10) directly rather than using the bound. N_{lr} is about 4×10^4, suggesting a lifetime for the GHZ state of roughly 8 ns. Using δt and $N_t = \log_\lambda N_{\rm lr}^{1/3}$, we find that constructing such a state would require a total time of about 25 ns.

To estimate the probability of performing state transfer instead of constructing the GHZ state, one must simply replace ϵ with $\sqrt{\epsilon}$ in the above analysis, as a state transfer success is effectively just two successful iterations of the GHZ construction. After state transfer is performed, we can ask whether it survives long enough to be read out or transferred into a non-interacting level. Since the single-atom lifetime of the Rydberg state is 340 μs, this should not be an issue as the time required to complete the transfer is on the order of tens of nanoseconds. Once transfer or GHZ creation is complete, the electric field can be turned off to remove the dipole-dipole interaction in Eq. (S1).

EFFECTS OF IMPERFECT SINGLE-QUBIT GATES

In addition to free evolution under the long-range interaction Hamiltonian [Eq. (S1)], our protocol requires a number of single-qubit gates to be performed. These can be Hadamard gates which produce the CNOT operation out of our ZZ interaction or the echoing pulses. In any case, a failure of the single-qubit gate can pose a serious problem to the protocol. Suppose we perform N_s single-qubit gates which succeed with a probability P. Then, as in the previous section, we demand that the gate sequence succeed with probability ϵ, obtaining

$$P^{N_s} > \epsilon \implies P > e^{(\ln \epsilon)/N_s}. \tag{S13}$$

The number of single qubit gates which must be targeted on a qubit in a timestep varies depending on that qubit's role during the step, but let us suppose that on average there are c gates per qubit performed on each of N_t timesteps. We can count the number of qubits involved in each timestep just as we did in Eq. (S8) to obtain a criterion for success:

$$P > e^{(\ln \epsilon)/(c\lambda^3(N-1)/(\lambda^3-1))}. \tag{S14}$$

Theoretical work on composite pulse sequences for atomic qubits suggests achievable fidelities of $1 - 10^{-4}$ [15]. If we assume $c = 4$ as an estimate, $\epsilon = 1/2$, and $\lambda = 2$ as in the last section, Eq. (S14) suggests that roughly 1500 qubits could be entangled with such gates using our protocol. This is a reduction of several orders of magnitude from the previous section which considered no single-qubit fidelity issues, a limitation which highlights the fact that a version of the protocol requiring less single-qubit control could perhaps entangle dramatically more qubits.

[1] M. Saffman, T. G. Walker, and K. Mølmer, Rev. Mod. Phys. **82**, 2313 (2010).

[2] D. Jaksch, J. I. Cirac, P. Zoller, S. L. Rolston, R. Côté, and M. D. Lukin, Phys. Rev. Lett. **85**, 2208 (2000).

[3] M. Saffman, J. Phys. B At. Mol. Opt. Phys. **49**, 202001 (2016).

[4] K. M. Maller, M. T. Lichtman, T. Xia, Y. Sun, M. J. Piotrowicz, A. W. Carr, L. Isenhower, and M. Saffman, Phys. Rev. A **92**, 022336 (2015).

[5] T. Wilk, A. Gaëtan, C. Evellin, J. Wolters, Y. Miroshnychenko, P. Grangier, and A. Browaeys, Phys. Rev. Lett. **104**, 010502 (2010).

[6] L. Isenhower, E. Urban, X. L. Zhang, A. T. Gill, T. Henage, T. A. Johnson, T. G. Walker, and M. Saffman, Phys. Rev. Lett. **104**, 010503 (2010).

[7] A. Browaeys, D. Barredo, and T. Lahaye, J. Phys. B At. Mol. Opt. Phys. **49**, 152001 (2016).

[8] A. V. Gorshkov, S. R. Manmana, G. Chen, E. Demler, M. D. Lukin, and A. M. Rey, Phys. Rev. A **84**, 033619 (2011).

[9] D. DeMille, Phys. Rev. Lett. **88**, 067901 (2002).

[10] B. Yan, S. A. Moses, B. Gadway, J. P. Covey, K. R. A. Hazzard, A. M. Rey, D. S. Jin, and J. Ye, Nature **501**, 521 (2013).

[11] M. A. Baranov, M. Dalmonte, G. Pupillo, and P. Zoller, Chem. Rev. **112**, 5012 (2012).

[12] J. I. Cirac and P. Zoller, Phys. Rev. Lett. **74**, 4091 (1995).

[13] Y. Wang, X. Zhang, T. A. Corcovilos, A. Kumar, and D. S. Weiss, Phys. Rev. Lett. **115**, 043003 (2015).

[14] B. Huber, T. Baluktsian, M. Schlagmüller, A. Kölle, H. Kübler, R. Löw, and T. Pfau, Phys. Rev. Lett. **107**, 243001 (2011).

[15] J. T. Merrill, S. C. Doret, G. Vittorini, J. P. Addison, and K. R. Brown, Phys. Rev. A **90**, 040301 (2014).