Supplemental material for "A solvable family of driven-dissipative many-body systems"

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This supplemental material contains an inductive proof of the localization of operators in the models considered, and also a proof that, under conditions specified in the manuscript and reiterated below, the Liouvillians we consider possesses a finite dissipative gap. In particular, the dissipative gap of the Liouvillian $\mathcal{L}$ is bounded below by that of the dissipator $\mathcal{D}$ in the absence of a Hamiltonian. The first section presents the proof by induction of Eq. (6) in the manuscript. The second section briefly introduces formalism that is helpful for proving the existence of a dissipative gap, the third section discusses the structure of the Liovillians considered in the manuscript in the language of this formalism, and the final section contains the proof of a dissipative gap.

A. Inductive proof of the localization of operators

In this section we begin with the time-series expansion of $O(t)$ given in the manuscript,

$$O(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \text{Tr}[\hat{\rho}_0 \hat{O}_n], \tag{S1}$$

and inductively prove the exact reduction to Eq. (6) of the manuscript,

$$O(t) = \text{Tr}_{\mathscr{A}\cup\mathscr{B}}\Big[\hat{O}\exp(\mathcal{L}_{\mathscr{A}\mathscr{B}})\hat{\rho}_{\mathscr{A}\mathscr{B}}\Big]. \tag{S2}$$

To begin, we decompose the Hamiltonian as $\hat{H} = \hat{H}_{\mathscr{A}\mathscr{B}} + \hat{H}_V$, where $\hat{H}_{\mathscr{A}\mathscr{B}}$ contains all terms in $\hat{H}$ that have support on $\mathscr{A}$, and $\hat{H}_V$ contains all terms that do not (note that, by the definition of $\mathscr{B}$, $\hat{H}_{\mathscr{A}\mathscr{B}}$ is supported on $\mathscr{A}\bigcup\mathscr{B}$, while $\hat{H}_V$ is supported on $\mathscr{B}\bigcup\mathscr{C}$). Similarly, we decompose the Heisenberg-picture dissipator as $\mathcal{D}^{\ddagger} = \mathcal{D}^{\ddagger}_{\mathscr{A}\mathscr{B}} + \mathcal{D}^{\ddagger}_V$, where $\mathcal{D}^{\ddagger}_{\mathscr{A}\mathscr{B}}$ only contains jump operators supported on $\mathscr{A}\bigcup\mathscr{B}$ and $\mathcal{D}^{\ddagger}_V$ only contains jump operators supported on $\mathscr{C}$. Writing $\mathcal{L}^{\ddagger} = \mathcal{L}^{\ddagger}_{\mathscr{A}\mathscr{B}} + \mathcal{L}^{\ddagger}_V$, with $\mathcal{L}^{\ddagger}_{\mathscr{A}\mathscr{B}}(\star) = i[\hat{H}_{\mathscr{A}\mathscr{B}}, \star] + \mathcal{D}^{\ddagger}_{\mathscr{A}\mathscr{B}}(\star)$ and $\mathcal{L}^{\ddagger}_V(\star) = i[\hat{H}_V, \star] + \mathcal{D}^{\ddagger}_V(\star)$, we can write $O(t)$ as

$$O(t) = \text{Tr}_{\mathscr{A}\cup\mathscr{B}}\Big[\text{Tr}_{\mathscr{C}}\Big[\hat{\rho}_0 \sum_{n=0}^{\infty}\frac{t^n}{n!}\hat{O}_n\Big]\Big], \tag{S3}$$

where $\hat{O}_n = (\mathcal{L}^{\ddagger}_{\mathscr{A}\mathscr{B}} + \mathcal{L}^{\ddagger}_V)^n\hat{O}$. The structure of Eq. (S3) can now be simplified by induction. Suppose that $\hat{O}_n$ satisfies the following two conditions:

(A) $\hat{O}_n$ is supported on $\mathscr{A}\bigcup\mathscr{B}$.

(B) $\hat{O}_n$ is diagonal on $\mathscr{B}$.

Note that both conditions are satisfied trivially for $\hat{O}_0 = \hat{O}$, since we have assumed that $\hat{O}$ is fully supported on $\mathscr{A}$. We will show that the operator $\hat{O}_{n+1} = (\mathcal{L}^{\ddagger}_{\mathscr{A}\mathscr{B}} + \mathcal{L}^{\ddagger}_V)\hat{O}_n$ also satisfies these conditions, thereby proving that all $\hat{O}_n$ do. Toward this end, we first show that $\mathcal{L}^{\ddagger}_V(\hat{O}_n) = i[\hat{H}_V, \hat{O}_n] + \mathcal{D}^{\ddagger}_V(\hat{O}_n)$ vanishes by the assumptions. The first term vanishes because $\hat{H}_V$ is diagonal and supported on $\mathscr{B}\bigcup\mathscr{C}$, while $\hat{O}_n$ is diagonal on $\mathscr{B}\bigcup\mathscr{C}$ by conditions (A,B), giving $[\hat{H}_V, \hat{O}_n] = 0$. The second term vanishes because $\mathcal{D}^{\ddagger}_V$ only contains jump operators supported on $\mathscr{C}$, implying (together with condition A) that $\mathcal{D}^{\ddagger}_V(\hat{O}_n) = 0$. Thus $\mathcal{L}^{\ddagger}_V(\hat{O}_n) = 0$ as claimed, giving $\hat{O}_{n+1} = \mathcal{L}^{\ddagger}_{\mathscr{A}\mathscr{B}}(\hat{O}_n)$. Now we note that, because $\hat{H}_{\mathscr{A}\mathscr{B}}$ and the jump operators in $\mathcal{D}^{\ddagger}_{\mathscr{A}\mathscr{B}}$ are supported on $\mathscr{A}\bigcup\mathscr{B}$, $\mathcal{L}^{\ddagger}_{\mathscr{A}\mathscr{B}}(\hat{O}_n)$ must also be supported on $\mathscr{A}\bigcup\mathscr{B}$, and $\hat{O}_{n+1}$ therefore satisfies condition (A). Furthermore, because $\mathcal{L}^{\ddagger}_{\mathscr{A}\mathscr{B}}$ is diagonality preserving, $\hat{O}_{n+1}$ continues to satisfy condition (B). Therefore, by induction, we deduce that $\hat{O}_n = (\mathcal{L}^{\ddagger}_{\mathscr{A}\mathscr{B}})^n\hat{O}$, and is supported on $\mathscr{A}\bigcup\mathscr{B}$. Plugging this result into Eq. (S3), using the isolated support of $\hat{O}_n$ to carry out the trace over $\mathscr{C}$, and defining $\rho_{\mathscr{A}\mathscr{B}} = \text{Tr}_{\mathscr{C}}[\hat{\rho}_0]$, we obtain

$$O(t) = \text{Tr}_{\mathscr{A}\cup\mathscr{B}}\Big[\hat{\rho}_{\mathscr{A}\mathscr{B}}\sum_{n=0}^{\infty}\frac{t^n}{n!}(\mathcal{L}^{\ddagger}_{\mathscr{A}\mathscr{B}})^n\hat{O}\Big]. \tag{S4}$$

Converting to the Schrödinger picture and resumming the series, we then obtain Eq. (S2).

B. Parametrization of density matrices and superoperators

An arbitrary density matrix of N spin-$\frac{1}{2}$'s can be expanded as

$$\hat{\rho} = \sum_{\{\mu_1,\dots,\mu_N\}} \rho_{\mu_1,\dots,\mu_N}\, \hat{\sigma}_1^{\mu_1} \otimes \cdots \otimes \hat{\sigma}_N^{\mu_N}; \qquad \mu_j \in \{1, x, y, z\}, \tag{S5}$$

where the coefficients $\rho_{\mu_1,\dots,\mu_N}$ are real numbers and $\hat{\sigma}_j^1 = \hat{\mathbb{1}}_j$ is the identity operator on site j. For convenience we also introduce a more compact notation in which the set of indices $\{\mu_1 \dots \mu_N\}$ is replaced by a single collective index μ (which runs over all possible assignments of values $1, x, y, z$ to all N sites), the basis element $\hat{\sigma}_1^{\mu_1} \otimes \cdots \otimes \hat{\sigma}_N^{\mu_N}$ is denoted simply by $\hat{\mu}$, and the density operator $\hat{\rho}$ (or any other operator) is denoted with a modified ket notation as $|\rho\rangle\rangle$. In this notation, Eq. (S5) becomes $|\rho\rangle\rangle = \sum_\mu \rho_\mu |\mu\rangle\rangle$. The ρ_μ must be real because the density matrix is Hermitian, and the basis elements all are Hermitian, but it is nevertheless useful to consider complex coefficients. In this notation a general (not-necessarily Hermitian) operator $\hat{A}$ and its Hermitian conjugate $\hat{A}^\dagger$ become, respectively, $|A\rangle\rangle = \sum_\mu A_\mu |\mu\rangle\rangle$ and $\langle\langle A| = \sum_\mu A_\mu^* \langle\langle \mu|$. The basis elements $|\mu\rangle\rangle$ form a vector space (Liouville space) endowed with the inner product $\langle\langle A|B\rangle\rangle \equiv 2^{-N}\mathrm{Tr}(\hat{A}^\dagger \hat{B}) = 2^{-N}\sum_{\mu\nu} A_\mu^* B_\nu \mathrm{Tr}(\hat{\mu}\hat{\nu}) = \sum_\mu A_\mu^* B_\mu$, where we have used the fact that $\hat{\mu}^\dagger = \hat{\mu}$ and $\mathrm{Tr}(\hat{\mu}\hat{\nu}) = 2^N \delta_{\mu,\nu}$. We denote the action of a superoperator on an operator, $\hat{B} = \mathcal{S}(\hat{A})$, with the notation $|B\rangle\rangle = \mathcal{S}|A\rangle\rangle$. The superoperator $\mathcal{S}$ inherets a matrix representation from the equation $|B\rangle\rangle = \mathcal{S}|A\rangle\rangle$ by expanding $|A\rangle\rangle = \sum_\nu A_\nu |\nu\rangle\rangle$ and left multiplying both sides of the equation by $\langle\langle\mu|$, giving $B_\mu = \sum_\nu \mathcal{S}_{\mu\nu} A_\nu$, where $\mathcal{S}_{\mu\nu} \equiv \langle\langle\mu|\mathcal{S}|\nu\rangle\rangle$. Note that a physical Liouvillian must be hermiticity preserving, which implies that its matrix elements in a basis of Hermitian operators (such as the operators $\hat{\mu}$ defined above) are always real.

C. Structure of the Liouvillian

For our purposes below, an important property of each basis element $|\mu\rangle\rangle$ is the number of operators $\hat{\sigma}_j^x$ and $\hat{\sigma}_j^y$ that appear [see Eq. (S5)], which we denote by d. In literature on nuclear-magnetic resonance Liouville space can be decomposed into a direct sum of subspaces with fixed d, and we denote the basis elements of these subspaces by $|d, \mu_d\rangle\rangle$, with the understanding that the index μ_d enumerates only the basis elements in this subspace. An arbitrary vector within this subspace is denoted by $|d, v\rangle\rangle = \sum_{\mu_d} v_{\mu_d} |d, \mu_d\rangle\rangle$. If we define $\mathcal{P}_d$ as a superoperator that projects onto the subspace spanned by $|d, \mu_d\rangle\rangle$, then we can expand $\mathcal{S} = \sum_{d,d'} \mathcal{P}_d \mathcal{S} \mathcal{P}_{d'} \equiv \sum_{d,d'} \mathcal{S}^{dd'}$, which has a matrix representation $\mathcal{S}^{dd'}_{\mu_d \nu_{d'}} = \langle\langle d, \mu_d|\mathcal{S}^{dd'}|d', \nu_{d'}\rangle\rangle$.

The Liouvillian superoperator can be decomposed as $\mathcal{L} = \mathcal{D} + i\mathcal{H}$, where $\mathcal{H}(\star) = -[\hat{H}, \star]$. Because $\mathcal{D}$ and $i\mathcal{H}$ are each (independently) hermiticity preserving, they both have real-valued matrix representations in the chosen basis. The dissipators we consider in the manuscript obey the condition $\mathrm{Tr}[\hat{\sigma}_j^z \mathcal{D}(\hat{\sigma}_j^\pm)] = \mathrm{Tr}[\hat{\mathbb{1}}_j \mathcal{D}(\hat{\sigma}_j^\pm)] = 0$ [Eq. (3) of the manuscript], which in the language used here implies that $\mathcal{D}^{dd'} = 0$ whenever $d < d'$. In fact, because the dissipator is constructed of local terms, it is straightforward to see that $\mathcal{D}^{dd'}$ vanishes unless either $d = d'$ or $d = d' + 1$. It is also readily verified that $\mathcal{H}$ is block diagonal in the d-subspace decomposition, and that $\mathcal{H}^{00} = 0$ (the latter condition following because $\hat{H}$ commutes with all basis elements in the $d = 0$ sector). Hence the complete Liouvillian has the block-lower-triangular structure (dropping subscripts)

$$\mathcal{L} = \begin{pmatrix} \mathcal{D}^{00} & 0 & 0 & \cdots & 0 & 0 \\ \mathcal{D}^{10} & \mathcal{D}^{11} + i\mathcal{H}^{11} & 0 & \cdots & 0 & 0 \\ 0 & \mathcal{D}^{21} & \mathcal{D}^{22} + i\mathcal{H}^{22} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \mathcal{D}^{N,N-1} & \mathcal{D}^{NN} + i\mathcal{H}^{NN} \end{pmatrix}. \tag{S6}$$

Note that the block-lower-triangular structure of $\mathcal{L}$ of ensures that the time evolution within the $d = 0$ subspace, which completely determines the populations in the z-basis, is governed by a closed set of equations. The dynamics within this sector is therefore *exactly* equivalent to the dynamics of a classical master equation. Note that a superficially similar situation can arise *approximately* in more general quantum systems in situations where the contribution of coherences can be perturbatively eliminated from the equations of motion for populations [1–5]. The eigenvalues of a block-lower triangular matrix are given by the eigenvalues of the diagonal blocks, and therefore

$$\mathrm{eigs}(\mathcal{L}) = \mathrm{eigs}(\mathcal{D}^{00}) \cup \mathrm{eigs}(\mathcal{D}^{11} + i\mathcal{H}^{11}) \cup \cdots \cup \mathrm{eigs}(\mathcal{D}^{NN} + i\mathcal{H}^{NN}). \tag{S7}$$

Here it is to be understood that the eigenvalues of a projected operator are computed only with respect to vectors supported on that subspace, i.e. the eigenvalues of $\mathcal{D}^{00}$ are given by the eigenvalues of the matrix $\mathcal{D}^{00}_{\mu_0,\nu_0}$.

D. Dissipative gap

The dissipators we consider are constructed as sums over single-site dissipators, $\mathcal{D} = \sum_j \mathcal{D}_j$, each of which independently obeys the constraints [equivalent to Eq. (3) in the manuscript]

$$\mathrm{Tr}[\hat{\mathbb{1}}_j \mathcal{D}_j(\hat{\sigma}^x_j)] = \mathrm{Tr}[\hat{\mathbb{1}}_j \mathcal{D}_j(\hat{\sigma}^y_j)] = \mathrm{Tr}[\hat{\sigma}^z_j \mathcal{D}_j(\hat{\sigma}^x_j)] = \mathrm{Tr}[\hat{\sigma}^z_j \mathcal{D}_j(\hat{\sigma}^y_j)] = 0. \tag{S8}$$

Here it is understood that $\mathcal{D}_j$ is an operator in the full Liouville space, but because it only acts nontrivially on site j we will (in a slight abuse of notation) temporarily use the symbol $\mathcal{D}_j$ to represent an operator acting on just the Liouville space associated with site j, spanned by vectors $|\mu_j\rangle\!\rangle$; regardless of the interpretation of which space $\mathcal{D}_j$ acts on, its eigenvalues are the same up to degeneracies. With this interpretation in mind, we can represent $\mathcal{D}_j$ as a 4×4 real matrix matrix $\langle\!\langle \mu_j|\mathcal{D}_j|\nu_j\rangle\!\rangle$. Using the shorthand $\mathcal{D}_{\mu\nu} \equiv \langle\!\langle \mu_j|\mathcal{D}_j|\nu_j\rangle\!\rangle$ (note the temporary suppression of the index j), ordering our basis as $\mu = 1, z, x, y$, and taking into account the constraints on $\mathcal{D}_j$ imposed in Eq. (S8), we have

$$\mathcal{D}_{\mu\nu} = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ \mathcal{D}_{z1} & \mathcal{D}_{zz} & 0 & 0 \\ \hline \mathcal{D}_{x1} & \mathcal{D}_{xz} & \mathcal{D}_{xx} & \mathcal{D}_{xy} \\ \mathcal{D}_{y1} & \mathcal{D}_{yz} & \mathcal{D}_{yx} & \mathcal{D}_{yy} \end{array}\right). \tag{S9}$$

In order to prove the existence of a dissipative gap for $\mathcal{L}$, we also make the following two assumptions:

(1) Each local dissipator $\mathcal{D}_j$ has a unique steady state and a dissipative gap Γ_j, and $\min_j \Gamma_j = \Gamma > 0$. Since all eigenvalues of the dissipator must have non-positive real parts, this means that for any nonzero eigenvalue λ of $\mathcal{D}_j$, we must have $\mathrm{Re}(\lambda) \le -\Gamma_j$.

(2) In addition to Eq. (S8), the local dissipators obey the further constraints:

$$\mathrm{Tr}[\hat{\sigma}^y_j \mathcal{D}_j(\hat{\sigma}^x_j)] = \mathrm{Tr}[\hat{\sigma}^x_j \mathcal{D}_j(\hat{\sigma}^y_j)], \quad \text{and} \quad \mathrm{Tr}[\hat{\sigma}^z_j \mathcal{D}_j(\hat{\mathbb{1}}_j)] = 0. \tag{S10}$$

In the notation just introduced, we can restate these conditions as: (1) The matrix $\mathcal{D}_{\mu\nu}$ has a unique zero eigenvalue and a dissipative gap Γ_j, and (2) The diagonal blocks of $\mathcal{D}_{\mu\nu}$ are symmetric ($\mathcal{D}_{z1} = 0$ and $\mathcal{D}_{yx} = \mathcal{D}_{xy}$). Note that the Hamiltonian in Eq. (9) of the manuscript, when rotated about the y axis and at $\Delta = 0$, becomes of the form in Eq. (1) of the manuscript, while the jump operators become $\hat{J}_j = \hat{\sigma}^y_j - i\hat{\sigma}^z_j$. It is straightforward to verify that conditions (1) and (2) are satisfied for the corresponding dissipator, with $\Gamma = \gamma$.

As mentioned above, a block-lower-triangular matrix of this form shares its eigenvalues with the diagonal blocks. Let's denote the set of eigenvalues of the upper-left block by $\mathscr{E}^0_j = \{0, \varepsilon^0\}$, and the set of eigenvalues of the lower-right block by $\mathscr{E}^1_j = \{\varepsilon^1_a, \varepsilon^1_b\}$ (it is straightforward to see that the upper-left block must have a zero eigenvalue). By conditions (1) and (2), we know that the three non-zero eigenvalues are real, and that $\varepsilon^0, \varepsilon^1_a, \varepsilon^1_b \le -\Gamma_j$. Because each $\mathcal{D}_j$ is supported on a single site, the eigenvalues of the full dissipator $\mathcal{D} = \sum_j \mathcal{D}_j$ are simply sums of eigenvalues from each $\mathcal{D}_j$. Eigenvalues of the projected dissipator $\mathcal{D}^{dd}$ can be constructed as follows: Defining $\mathscr{J}_d$ to be a set containing d sites, and also defining the notation $\mathscr{A} + \mathscr{B} = \{a + b :\ a \in \mathscr{A},\ b \in \mathscr{B}\}$, we have

$$\mathrm{eigs}(\mathcal{D}^{dd}) = \bigcup_{\mathscr{J}_d} \Big(\sum_{j \in \mathscr{J}_d} \mathscr{E}^1_j + \sum_{j \notin \mathscr{J}_d} \mathscr{E}^0_j \Big). \tag{S11}$$

We can draw two immediate conclusions:

(A) $\mathcal{D}^{00}$ has precisely one zero eigenvalue, obtained when the zero eigenvalue in $\mathscr{E}^0_j$ is chosen for all j, and any other eigenvalue ε satisfies $\varepsilon \le -\Gamma$.

(B) Any eigenvalue ε of $\mathcal{D}^{dd}$ (with $d > 0$) is real and satisfies $\varepsilon < -d\Gamma$.

Now consider an arbitrary normalized eigenvector of $\mathcal{L}^{dd}$ in the $d \neq 0$ subspace, $|d, \nu\rangle\!\rangle$, with eigenvalue $\varepsilon = x + iy$, and write

$$\langle\!\langle d, \nu|\mathcal{L}^{dd}|d, \nu\rangle\!\rangle = \langle\!\langle d, \nu|\mathcal{D}^{dd}|d, \nu\rangle\!\rangle + i\langle\!\langle d, \nu|\mathcal{H}^{dd}|d, \nu\rangle\!\rangle = x + iy. \tag{S12}$$

It is straightforward to show that the superoperator $\mathcal{H}^{dd}$ is Hermitian (for any Hamiltonian), and thus $x = \mathrm{Re}(\langle\!\langle d, \nu|\mathcal{D}^{dd}|d, \nu\rangle\!\rangle)$. In general, there is not a simple relation between the real part of the expectation value of an operator and the real parts of its eigenvalues. However, assumption (2) guarantees that $\mathcal{D}^{dd}$ is a real symmetric matrix. Therefore, we have

$$x = \langle\!\langle d, \nu|\mathcal{D}^{dd}|d, \nu\rangle\!\rangle \le -d\Gamma, \tag{S13}$$

where the inequality follows because the expectation value of a *real symmetric matrix* (more generally a Hermitian matrix) is bounded between its smallest and largest eigenvalues. Returning to Eq. (S7), keeping in mind conclusion (A), and using Eq. (S13), we find that $\mathcal{L}$ has a single zero eigenvalue, and a dissipative gap of at least Γ.

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