

Asymmetric Particle Transport and Light-Cone Dynamics Induced by Anyonic Statistics (Supplemental Material)

Fangli Liu,¹ James R. Garrison,^{1,2} Dong-Ling Deng,^{3,4,1} Zhe-Xuan Gong,^{5,1,2} and Alexey V. Gorshkov^{1,2}

¹*Joint Quantum Institute, NIST/University of Maryland, College Park, Maryland 20742, USA*

²*Joint Center for Quantum Information and Computer Science,
NIST/University of Maryland, College Park, Maryland 20742, USA*

³*Center for Quantum Information, IIIS, Tsinghua University, Beijing 100084, PR China*

⁴*Condensed Matter Theory Center, Department of Physics,
University of Maryland, College Park, Maryland 20742, USA*

⁵*Department of Physics, Colorado School of Mines, Golden, Colorado 80401, USA*

This Supplemental Material consists of three sections. In Sec. **S.I**, we derive the dynamical symmetry given by Eqs. (9) and (10) in the main text. In Sec. **S.II**, we provide an intuitive derivation of the asymmetric expansion dynamics, based on perturbation theory. In Sec. **S.III**, we compare features of the bosonic OTOC (which is experimentally accessible) to the anyonic OTOC given by Eq. (11) in the main text.

S.I. DYNAMICAL SYMMETRY OF DENSITY EXPANSION

In this section, we give detailed derivations for the dynamical symmetry observed in the main text in Eqs. (9) and (10). The inversion symmetry operator \mathcal{I} acts on a bosonic operator as $\mathcal{I}\hat{b}_j\mathcal{I}^\dagger = \hat{b}_{j'}$, where j' is the site that j is mapped to under reflection about the middle of the 1D system. The time-reversal operator \mathcal{T} acts by complex-conjugating the entries of a state (or operator) written in the bosonic Fock basis; for instance, $\mathcal{T}\hat{b}_j\mathcal{T}^{-1} = \hat{b}_j$ and $\mathcal{T}i\mathcal{T}^{-1} = -i$. Although \hat{H}_B respects neither time-reversal nor inversion symmetry, it does obey the following \mathcal{K} symmetry [S?]:

$$\mathcal{K}\hat{H}_B\mathcal{K}^\dagger = \hat{H}_B, \quad (\text{S1})$$

where $\mathcal{K} = \mathcal{R}\mathcal{I}\mathcal{T}$, and \mathcal{R} is defined as

$$\mathcal{R} = e^{-i\theta \sum_j \hat{n}_j (\hat{n}_j - 1)/2}. \quad (\text{S2})$$

With this, we now consider the symmetry properties of the particle dynamics. Using Eq. (S1), one has:

$$\mathcal{K}e^{-i\hat{H}_B t}\mathcal{K}^\dagger = e^{i\hat{H}_B t}, \quad (\text{S3})$$

where we have used the anti-unitary property of the \mathcal{K} operator. We first focus on the symmetry properties when flipping the sign of θ [Eq. (10) in the main text]. We label \hat{H}_B with the sign of θ for convenience:

$$\hat{H}_{B,\pm\theta} = -J \sum_{j=1}^{L-1} \left(\hat{b}_j^\dagger \hat{b}_{j+1} e^{\pm i\theta \hat{n}_j} + \text{H.c.} \right) + \frac{U}{2} \sum_{j=1}^L \hat{n}_j (\hat{n}_j - 1). \quad (\text{S4})$$

The time-dependent density at site j is

$$\langle \hat{n}_j(t) \rangle_{\pm\theta} \equiv \langle \Psi_0 | e^{i\hat{H}_{B,\pm\theta} t} \hat{n}_j e^{-i\hat{H}_{B,\pm\theta} t} | \Psi_0 \rangle, \quad (\text{S5})$$

where $|\Psi_0\rangle$ is the initial Fock product state given in the main text, $|\Psi_0\rangle = \prod_i \hat{b}_i^\dagger |0\rangle$. (We have omitted the subscript “ B ” for simplicity.) We obtain

$$\begin{aligned} \langle \hat{n}_j(t) \rangle_{+\theta} &\equiv \langle \Psi_0 | e^{i\hat{H}_{B,+\theta} t} \hat{n}_j e^{-i\hat{H}_{B,+\theta} t} | \Psi_0 \rangle \\ &= \langle \Psi_0 | \mathcal{K}^\dagger e^{-i\hat{H}_{B,+\theta} t} \mathcal{K} \hat{n}_j \mathcal{K}^\dagger e^{i\hat{H}_{B,+\theta} t} \mathcal{K} | \Psi_0 \rangle \\ &= \langle \Psi_0 | e^{-i\hat{H}_{B,+\theta} t} \mathcal{I} \hat{n}_j \mathcal{I}^\dagger e^{i\hat{H}_{B,+\theta} t} | \Psi_0 \rangle \\ &= \langle \Psi_0 | e^{-i\hat{H}_{B,+\theta} t} \hat{n}_{j'} e^{i\hat{H}_{B,+\theta} t} | \Psi_0 \rangle, \end{aligned} \quad (\text{S6})$$

where, in the second line, we have sandwiched $\mathcal{K}^\dagger\mathcal{K}$ between each two operators and used Eq. (S3); in the third line, we have used (i) the fact that when \mathcal{K} operates on the initial state $|\Psi_0\rangle$ in the main text, it gives an unimportant phase after complex conjugation, and (ii) the relation $\mathcal{K}\hat{n}_j\mathcal{K}^\dagger = \mathcal{I}\hat{n}_j\mathcal{I}^\dagger$; in the fourth line, we have defined the density operator $\hat{n}_{j'}$ on site j' , which is related to \hat{n}_j by the inversion symmetry operator \mathcal{I} .

To proceed, we relate $\hat{H}_{B,\pm\theta}$ by the time-reversal symmetry operator \mathcal{T} :

$$\mathcal{T}\hat{H}_{B,+\theta}\mathcal{T}^{-1} = \hat{H}_{B,-\theta}. \quad (\text{S7})$$

Thus,

$$\mathcal{T}e^{-i\hat{H}_{B,+\theta} t}\mathcal{T}^{-1} = e^{i\hat{H}_{B,-\theta} t}. \quad (\text{S8})$$

Substituting the above equation into Eq. (S6), we get:

$$\begin{aligned} \langle \hat{n}_j(t) \rangle_{+\theta} &= \langle \Psi_0 | e^{-i\hat{H}_{B,+\theta} t} \hat{n}_{j'} e^{i\hat{H}_{B,+\theta} t} | \Psi_0 \rangle \\ &= \langle \Psi_0 | \mathcal{T}^{-1} e^{i\hat{H}_{B,-\theta} t} \mathcal{T} \hat{n}_{j'} \mathcal{T}^{-1} e^{-i\hat{H}_{B,-\theta} t} | \Psi_0 \rangle \\ &= \langle \Psi_0 | e^{i\hat{H}_{B,-\theta} t} \hat{n}_{j'} e^{-i\hat{H}_{B,-\theta} t} | \Psi_0 \rangle \\ &\equiv \langle \hat{n}_{j'}(t) \rangle_{-\theta}. \end{aligned} \quad (\text{S9})$$

Finally, we arrive at a very simple equation [Eq. (10) in the main text]: $\langle \hat{n}_j(t) \rangle_{+\theta} = \langle \hat{n}_{j'}(t) \rangle_{-\theta}$. This relation just tells us that when flipping the statistical angle θ , the density expectation values are related by inversion, which agrees with our results in Figs. 1(f) and (g) in the main text. For $\theta = 0$ or π , we have $\langle \hat{n}_j(t) \rangle_{0,+\pi} = \langle \hat{n}_{j'}(t) \rangle_{0,-\pi} = \langle \hat{n}_{j'}(t) \rangle_{0,+\pi}$; that is, for the boson case ($\theta = 0$) or the pseudofermion case ($\theta = \pi$), the density expands symmetrically whether or not $U = 0$.

There remains another dynamical symmetry [Eq. (9) in the main text]: when changing the sign of the interaction U , one gets the same behavior as changing the sign of θ , i.e., the two density expansions are related by inversion symmetry. Let us now derive this relation.

Like in Eq. (S4), we label \hat{H}_B with the sign of U :

$$\hat{H}_{B,\pm U} = -J \sum_{j=1}^L \left(\hat{b}_j^\dagger \hat{b}_{j+1} e^{i\theta \hat{n}_j} + \text{H.c.} \right) \pm \frac{U}{2} \sum_{j=1}^L \hat{n}_j (\hat{n}_j - 1). \quad (\text{S10})$$

Replacing $\hat{H}_{B,+ \theta}$ with $\hat{H}_{B,+U}$ in Eq. (S6), we get

$$\langle \hat{n}_j(t) \rangle_{+U} = \langle \Psi_0 | e^{-i\hat{H}_{B,+U}t} \hat{n}_{j'} e^{i\hat{H}_{B,+U}t} | \Psi_0 \rangle. \quad (\text{S11})$$

Now let us define a number parity operator, $\mathcal{P} = e^{i\pi \sum_r \hat{n}_{2r+1}}$, which measures the parity of total particle number on the odd sites. This operator anti-commutes with the first term of Eq. (S10), but commutes with the second term. Therefore,

$$\begin{aligned} \mathcal{P} \hat{H}_{B,+U} \mathcal{P}^\dagger &= J \sum_{j=1}^L \left(\hat{b}_j^\dagger \hat{b}_{j+1} e^{i\theta \hat{n}_j} + \text{H.c.} \right) + \frac{U}{2} \sum_{j=1}^L \hat{n}_j (\hat{n}_j - 1) \\ &\equiv -\hat{H}_{B,-U}. \end{aligned} \quad (\text{S12})$$

Thus,

$$\mathcal{P} e^{-i\hat{H}_{B,+U}t} \mathcal{P}^\dagger = e^{i\hat{H}_{B,-U}t}. \quad (\text{S13})$$

Substituting the above equation into Eq. (S11) results in

$$\begin{aligned} \langle \hat{n}_j(t) \rangle_{+U} &= \langle \Psi_0 | e^{-i\hat{H}_{B,+U}t} \hat{n}_{j'} e^{i\hat{H}_{B,+U}t} | \Psi_0 \rangle \\ &= \langle \Psi_0 | \mathcal{P}^\dagger e^{i\hat{H}_{B,-U}t} \mathcal{P} \hat{n}_{j'} \mathcal{P}^\dagger e^{-i\hat{H}_{B,-U}t} | \Psi_0 \rangle \\ &= \langle \Psi_0 | e^{i\hat{H}_{B,-U}t} \hat{n}_{j'} e^{-i\hat{H}_{B,-U}t} | \Psi_0 \rangle \\ &= \langle \hat{n}_{j'}(t) \rangle_{-U}. \end{aligned} \quad (\text{S14})$$

Once again, we arrive at a simple expression [Eq. (9) in the main text], $\langle \hat{n}_j(t) \rangle_{+U} = \langle \hat{n}_{j'}(t) \rangle_{-U}$, which confirms that by changing the sign of interaction U , the density expansion of anyons undergoes an inversion operation. For zero interaction strength, we have $\langle \hat{n}_j(t) \rangle_{U=+0} = \langle \hat{n}_{j'}(t) \rangle_{U=-0} = \langle \hat{n}_{j'}(t) \rangle_{U=+0}$. Therefore, the density expansion of anyons is symmetric when $U = 0$, regardless of whether θ is a multiple of π .

More generally, it is straightforward to show that the dynamical symmetry relations shown in Eqs. (S9) and (S14) hold for a class of initial states satisfying $\mathcal{K}|\Psi\rangle = e^{i\phi}(|\Psi\rangle)^*$ for some ϕ .

S.II. PERTURBATION ANALYSIS OF ASYMMETRIC EXPANSION

In this section, we provide intuition while deriving the asymmetric expansion using perturbation theory. Specifically,

we show that the interference between the lowest two order terms in the unitary evolution generally gives rise to asymmetric density expansion dynamics. Once again, we focus on the transformed bosonic Hamiltonian (\hat{H}_B) for simplicity.

Using a Taylor expansion, the unitary time evolution operator can be written as

$$\mathcal{U} = e^{-i\hat{H}_B t} = \sum_{n=0}^{\infty} \frac{(-i\hat{H}_B t)^n}{n!} = 1 - i\hat{H}_B t + \frac{(i\hat{H}_B t)^2}{2!} - \dots \quad (\text{S15})$$

We assume the initial state $|\psi_0\rangle$ to be a product state (in Fock space) that is inversion symmetric around the lattice center (i.e., $\mathcal{I}|\psi_0\rangle = |\psi_0\rangle$). The final state after time evolution can be expanded as a sum of product states in Fock space. We consider, as target states, a pair of such product states which are related by inversion symmetry, $|\psi_2\rangle = \mathcal{I}|\psi_1\rangle$, and show that their overlaps with the time-evolved state are different due to the interference of the k th and $(k+1)$ th order terms in the expansion.

We denote the matrix element corresponding to the k th order term evolving $|\psi_0\rangle$ to $|\psi_1\rangle$ as

$$M_k^{(1)} = \left\langle \psi_1 \left| \frac{(-i\hat{H}_B t)^k}{k!} \right| \psi_0 \right\rangle = \frac{(-it)^k}{k!} A_k, \quad (\text{S16})$$

where we have defined $A_k = \langle \psi_1 | \hat{H}_B^k | \psi_0 \rangle$. Similarly, $M_k^{(2)}$ is the matrix element from $|\psi_0\rangle$ to $|\psi_2\rangle$ due to the k th order term:

$$M_k^{(2)} = \left\langle \psi_2 \left| \frac{(-i\hat{H}_B t)^k}{k!} \right| \psi_0 \right\rangle = \frac{(-it)^k}{k!} B_k, \quad (\text{S17})$$

where $B_k = \langle \psi_2 | \hat{H}_B^k | \psi_0 \rangle$. Using the symmetry properties of the Hamiltonian, we can get:

$$\begin{aligned} B_k &= \langle \psi_2 | \hat{H}_B^k | \psi_0 \rangle = \langle \psi_1 | \mathcal{I}^\dagger \hat{H}_B^k \mathcal{I} | \psi_0 \rangle \\ &= e^{i(\phi_2 - \phi_0)} \langle \psi_1 | \mathcal{I}^\dagger \mathcal{R}^\dagger \hat{H}_B^k \mathcal{R} \mathcal{I} | \psi_0 \rangle \\ &= e^{i(\phi_2 - \phi_0)} \langle \psi_1 | (\mathcal{T} \hat{H}_B^k \mathcal{T}^{-1}) | \psi_0 \rangle \\ &= e^{i(\phi_2 - \phi_0)} (\langle \psi_1 | \hat{H}_B^k | \psi_0 \rangle)^* \\ &= e^{i(\phi_2 - \phi_0)} A_k^*, \end{aligned} \quad (\text{S18})$$

where in the second line, we have used the symmetry relation between $|\psi_1\rangle$ and $|\psi_2\rangle$ and the fact that $|\psi_0\rangle$ is symmetric under \mathcal{I} ; in the third line, we extract the phase factor associated with the action of the \mathcal{R} symmetry operator [defined in Eq. (S2)] on states $|\psi_{0,2}\rangle$: $\mathcal{R}|\psi_{0,2}\rangle = e^{i\phi_{0,2}}|\psi_{0,2}\rangle$; in the fourth line, we have used the symmetry property given by Eq. (S1); and in the fifth line, we have used the fact that the time-reversal operator acting on \hat{H}_B is equivalent to changing the matrix element to its complex conjugate.

From here forward, let k be the lowest order for which $M_k^{(1)}$ [or, equivalently, $M_k^{(2)}$] is non-zero. Because the Hamiltonian \hat{H}_B can have non-zero interactions U , the $(k+1)$ th expansion terms could also evolve the initial state to $|\psi_{1,2}\rangle$. Therefore,

we consider the leading two order terms which contribute to the matrix element for $\langle \psi_{1,2} | \mathcal{U} | \psi_0 \rangle$: $M_k^{(1,2)}$ and $M_{k+1}^{(1,2)}$. We define $S_{1,2}$ to be amplitudes including the total contribution of the k th and $(k+1)$ th orders:

$$S_1 = \left| M_k^{(1)} + M_{k+1}^{(1)} \right| = \frac{t^k}{k!} \left| A_k + \frac{-it}{k+1} A_{k+1} \right|, \quad (\text{S19})$$

$$S_2 = \left| M_k^{(2)} + M_{k+1}^{(2)} \right| = \frac{t^k}{k!} \left| B_k + \frac{-it}{k+1} B_{k+1} \right|. \quad (\text{S20})$$

Using Eq. (S18), Eq. (S20) can be re-written as

$$\begin{aligned} S_2 &= \frac{t^k}{k!} \left| B_k + \frac{-it}{k+1} B_{k+1} \right| = \frac{t^k}{k!} \left| A_k^* + \frac{-it}{k+1} A_{k+1}^* \right| \\ &= \frac{t^k}{k!} \left| A_k - \frac{-it}{k+1} A_{k+1} \right|. \end{aligned} \quad (\text{S21})$$

Comparing Eqs. (S19) and (S21), we can see that because the sign before A_{k+1} is different, the two amplitudes S_1 and S_2 are in general not equal to each other. This is a simple way of understanding the observed asymmetric expansion in the left and right directions.

The following remarks regarding S_1 and S_2 are in order: (i) If we set $\theta = 0$ or $\theta = \pi$, the matrix elements A_k and A_{k+1} are both real numbers. In this case, S_1 and S_2 are exactly equal to each other. This implies that for zero statistical angle θ , the perturbation analysis predicts symmetric density expansion, consistent with our numerics. (ii) On the other hand, for non-zero θ , A_k and A_{k+1} are generally complex numbers, and S_1 and S_2 are not necessarily equal, therefore predicting asymmetric expansion in general. (iii) When θ reverses its sign, all the matrix elements change to their complex conjugates, and therefore the values of S_1 and S_2 are swapped. In this way, the anyons reverse their preferred propagation directions, in agreement with the numerical results. (iv) When the interaction strength U is zero, the matrix element $M_{k+1}^{(1)}$ vanishes, since the Hamiltonian only has hopping terms and hopping once more could not get back to the same state configuration as $|\psi_{1,2}\rangle$. Therefore, S_1 and S_2 are the same when $U = 0$. (v) When U 's sign is reversed, A_{k+1} also reverses its sign, therefore swapping the values of S_1 and S_2 . Thus, the anyons once again reverse their preferred propagation directions.

The above analysis is completely consistent with the numerical results in the main text. We have once again demonstrated that the crucial ingredients for asymmetric expansion are non-zero statistics θ and interaction U . To illustrate more clearly the above derivations, we consider a very simple example for clarification. Let us choose $|\psi_0\rangle = |\cdots 0110 \cdots\rangle$, $|\psi_1\rangle = |\cdots 0011 \cdots\rangle$, $|\psi_2\rangle = |\cdots 1100 \cdots\rangle$. In this case, the second- and third-order terms in the perturbative time evolution could evolve $|\psi_0\rangle$ to $|\psi_1\rangle$ if U is non-zero. For second-order processes, there are two paths one can start from $|\psi_0\rangle$ and end up with $|\psi_1\rangle$: either $|\cdots 0110 \cdots\rangle \rightarrow |\cdots 0101 \cdots\rangle \rightarrow |\cdots 0011 \cdots\rangle$ or

$|\cdots 0110 \cdots\rangle \rightarrow |\cdots 0020 \cdots\rangle \rightarrow |\cdots 0011 \cdots\rangle$. The two paths contribute to a total second-order matrix element $\langle \psi_1 | \hat{H}_B^2 | \psi_0 \rangle = J^2 + J^2 e^{i\theta}$. Due to the on-site interactions, there is also a third-order process which evolves $|\psi_0\rangle$ to $|\psi_1\rangle$: $|\cdots 0110 \cdots\rangle \rightarrow |\cdots 0020 \cdots\rangle \rightarrow |\cdots 0020 \cdots\rangle \rightarrow |\cdots 0011 \cdots\rangle$, whose matrix element is $\langle \psi_1 | \hat{H}_B^3 | \psi_0 \rangle = J^2 U e^{i\theta}$. The total amplitude for second and third order processes is $S_1 = \frac{t^2}{2} |J^2(1 + e^{i\theta}) + \frac{-it}{3} J^2 U e^{i\theta}|$. Similarly we can also obtain $S_2 = \frac{t^2}{2} |J^2(1 + e^{-i\theta}) + \frac{-it}{3} J^2 U e^{-i\theta}|$. For non-zero θ and U , $S_1 \neq S_2$, implying asymmetric expansion. The expressions also predict that the expansion changes its preferred direction when either θ or U reverses its sign.

S.III. NUMERICAL COMPARISON OF ANYONIC AND BOSONIC OUT-OF-TIME-ORDERED CORRELATORS

In this section, we provide numerical results for the bosonic OTOC, $\tilde{F}_{jk}(t) = \langle \hat{b}_j^\dagger(t) \hat{b}_k^\dagger(0) \hat{b}_j(t) \hat{b}_k(0) \rangle$, to illustrate that such experimentally measurable quantities can indeed capture the asymmetric information spreading.

Figure S1 shows the bosonic OTOC growth, with parameters the same as Fig. 3 in the main text. As one can see, the bosonic OTOCs with non-zero statistical angle also exhibit asymmetric information propagation, similar to their anyonic counterparts.

Figure S2 shows the butterfly velocities extracted from the bosonic OTOC. In order to make comparisons to anyonic results, we also plot data from Figs. 4(c) and (d) of the main text. As the figures illustrate, the bosonic butterfly velocities are highly asymmetric for the left and right propagation directions. Moreover, in the regimes of either small θ or large U , both the left and right velocities of the bosonic OTOC agree well with the anyonic OTOC. This can be understood intu-

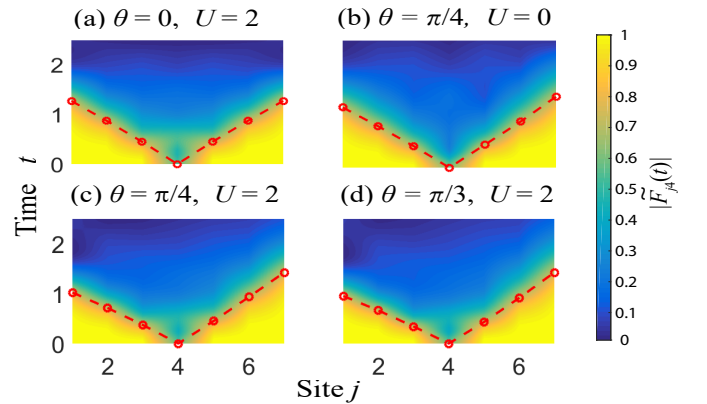


FIG. S1. Growth of the bosonic OTOC $|\tilde{F}_{jk}(t)|$ for different statistical angles θ and interaction strengths U . (a) Bosonic case ($\theta = 0$) with interaction strength $U = 2$. Anyonic case with (b) vanishing and (c),(d) non-vanishing interaction strengths. As in Fig. 3, $L = 7$, $\beta^{-1} = 6$, the local Hilbert space of each site is truncated to three states, and the red dots denote the OTOC falling to 75% of its initial value.

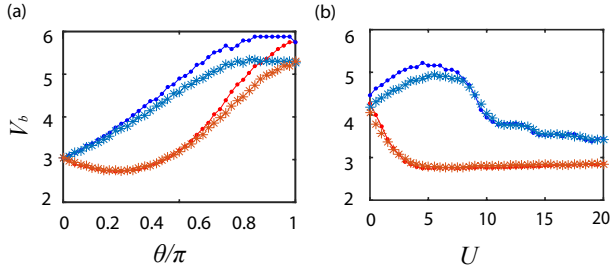


FIG. S2. Comparison of butterfly velocities extracted from the anyonic (dots) and bosonic (asterisks) OTOCs' growth. (a) The butterfly velocities' dependence on statistical angle θ for fixed $U = 2$. The blue dots/asterisks denote the butterfly velocities in the left direction, while the red dots/asterisks denote the butterfly velocities in the right direction. (b) Similar to (a), but for fixed statistical angle $\theta = \pi/2$ and varying interaction strength U .

itively, as the fractional Jordan-Wigner transformation has reduced effect at small θ , and large U corresponds to the hardcore limit, where anyonic statistics becomes less important. Moreover, the bosonic/anyonic plots in Fig. S2 share qualitative features for all values of θ or U . This suggests that the bosonic OTOC also exhibits signatures of the asymmetric propagation of information due to anyonic statistics.

[S] F. Lange, S. Ejima, and H. Fehske, "Anyonic Haldane Insulator in One Dimension," *Phys. Rev. Lett.* **118**, 120401 (2017).