Parafermionic Zero Modes in Ultracold Bosonic Systems

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Exotic topologically protected zero modes with parafermionic statistics (also called fractionalized Majorana modes) have been proposed to emerge in devices fabricated from a fractional quantum Hall system and a superconductor. The fractionalized statistics of these modes takes them an important step beyond the simplest non-Abelian anyons, Majorana fermions. Building on recent advances towards the realization of fractional quantum Hall states of bosonic ultracold atoms, we propose a realization of parafermions in a system consisting of Bose-Einstein-condensate trenches within a bosonic fractional quantum Hall state. We show that parafermionic zero modes emerge at the end points of the trenches and give rise to a topologically protected degeneracy. We also discuss methods for preparing and detecting these modes.

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In recent years the concept of topological order has revolutionized the way we understand quantum phases of matter. Topological phases in one- and two-dimensional systems are particularly interesting as the nontrivial exchange statistics of particles allows for exotic states of matter. For example, Majorana zero modes can emerge at boundaries of one-dimensional topological superconductors [1] or in two-dimensional semiconductor heterostructures [2–4]; see also suggestive experimental signatures in Refs. [5–10]. These topological modes have been a subject of intense interest due to their potential applications in quantum computation, although they do not support universal quantum computing with braiding alone [11–16]. In two dimensions, certain fractional quantum Hall (FQH) states have been proposed to manifest emergent non-Abelian excitations with universal braiding statistics [17–19]. However, an experimental confirmation of such emergent non-Abelian anyons has so far remained elusive [20–22].

Recently, it has been proposed that one can engineer non-Abelian excitations by adding defects in the form of ferromagnet-superconductor interfaces at the edges of adjacent Abelian FQH states [23–26]. The domain wall at their interface binds exotic zero modes with parafermionic commutation relations. These parafermionic zero modes are associated with a ground-state degeneracy that is exponential in the number of domain walls (defects). As with any non-Abelian anyon, the exchange of the defects binding these parafermionic operators generates a unitary rotation in the ground-state subspace. However, the set of operations for quantum computation available through such exchanges is richer than that available through Majorana exchange [23]. While the braiding of parafermions is not in itself universal for quantum computation, recent proposals have also used these modes in constructing new 2D topological phases that do support excitations with computationally universal braid statistics [27]. Parafermions may also be realized in bilayer quantum Hall systems, where the role of the superconducting-induced coupling is played by an interlayer tunneling term [28,29]. For a recent proposal on parafermions, see also Ref. [30].

While existing proposals are based on an experimentally challenging combination of FQH and superconducting systems of electrons, rapid advances towards creating a bosonic FQH state open new opportunities to realize topologically nontrivial states in the context of ultracold bosonic systems [31–34]. FQH states have been predicted for ultracold neutral bosons in rotating traps [35–39] and in optical lattices [40–44].

Unlike fermionic systems, where a condensed state requires pairing of fermions, systems of ultracold bosons form Bose-Einstein condensates (BECs) without any additional pairing interaction. Thus, if a FQH state is realized in bosons, adding a Bose condensed state to such a system can be expected to be simpler than in the corresponding fermionic implementation.

In this Letter, we propose realizing a bosonic analog of a FQH-superconductor system by inserting a trench containing a BEC in the middle of a bosonic FQH system, as shown in Fig. 1. Such a trench of BEC can be created by introducing a potential well that would trap a high density of bosons as compared to the FQH region outside. We will show that the introduction of such a BEC trench induces a novel state in the quantum Hall edge with a pairing gap similar to the states with superconductors in contact with the FQH system. Further, we will show that systems containing a pair of such trenches feature a ground state degeneracy arising from the topology of the underlying quantum Hall system. We explicitly construct fractionalized Majorana fermions as localized zero modes at the end points of the trenches.

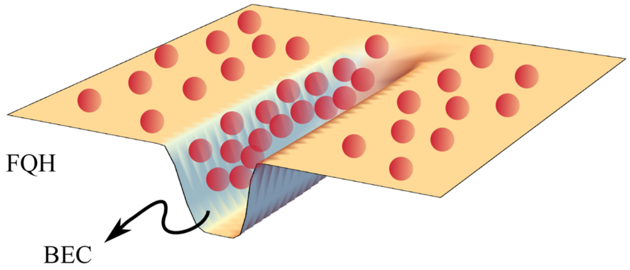

FIG. 1 (color online). A quasi-one-dimensional finite trench, i.e., a potential dip, is created by spatially modulating the intensity of the laser used to create the dipole trapping potential confining the atoms to the plane (a cut midway along one trench is shown). Within the trench, the two-dimensional density of trapped bosons deviates from the FQH filling fraction and gives rise to a BEC.

Model.—Let us consider a bosonic FQH state with a filling fraction $\nu = 1/m$, where m is an even integer. While the bulk of the FQH fluid is incompressible, its boundaries support gapless edge excitations with a fractionalized charge, or boson number in this case, and statistics [45,46]. The edge modes of a FQH state can be described by a bosonized field $e^{i\varphi(\xi)}$ that carries boson number $1/m$ and satisfies a nontrivial commutation relation [47]

$$e^{i\varphi(\xi)}e^{i\varphi(\xi')} = e^{i(\pi/m)\mathrm{sgn}(\xi-\xi')}e^{i\varphi(\xi')}e^{i\varphi(\xi)}, \quad (1)$$

where ξ and ξ' are the coordinates along the boundary. Equivalently, the chiral field φ satisfies the chiral commutation relations. The creation operator for bosons is given by $e^{im\varphi(\xi)}$; note that bosonic operators commute according to the above algebra. Furthermore, the density of bosons on the edge is given by $\rho(\xi) = \partial_\xi\varphi/2\pi$. The Hamiltonian $H_0 = (mv/4\pi)\int_\Gamma d\xi(\partial_\xi\varphi)^2$, along with the chiral commutation $[\varphi(\xi), \partial_{\xi'}\varphi(\xi')] = (2\pi i/m)\delta(\xi-\xi')$ describes a free chiral edge mode propagating at velocity v along the boundary Γ of the FQH state.

As shown in Fig. 1, we consider a BEC trench inside a bosonic FQH state that can be introduced by varying the potential. The topological phase realized by this FQH system will be characterized by a topological degeneracy. To create a Hilbert space that allows us to access this degeneracy, we will need to consider a system containing two such trenches in the FQH system [two copies of the trench in Fig. 1]. The total Hamiltonian describing the boundaries of the FQH state at the edges of the two BEC trenches is given by

$$H = H_1(\varphi_1, Q_1) + H_2(\varphi_2, Q_2)$$
$$\text{with} \quad Q_1 + Q_2 = 0 \mod 1, \quad (2)$$

where the subscripts denote the trenches, and Q_i is the fractional number ("charge") of quasiparticles modulo 1 on the edge of the ith trench. While each edge can have a fractional charge, the total number of quasiparticles must add up to an integer bosonic charge, i.e., $Q_1 + Q_2 = 0 \bmod 1$. We show that, under certain conditions, the Hilbert space of this system manifests degenerate ground states characterized by parafermionic zero modes at the ends of the BEC trenches. The defects at each end of the trench thus act as non-Abelian anyons. This is the central result of this Letter. We also analyze the robustness of this degeneracy to realistic experimental imperfections.

Single-trench Hamiltonian.—For simplicity, we first focus on a single edge, with the Hamiltonian $H_1(\varphi_1, Q_1)$, without the constraint in Eq. (2), but will impose it later in our discussion of the degeneracy of the two-trench system. For now, we also drop the subscript 1 for notational simplicity. The tunneling between the BEC field Ψ and the edge states on the trench is described by

$$H_{\mathrm{tun}} = -\Delta\int_\Gamma d\xi e^{im\varphi(\xi)}\Psi(\xi) + \mathrm{H.c.} \quad (3)$$

Note that only a boson, i.e., m quasiparticles bound together, can directly tunnel to the BEC. Opening a topological gap on the trench requires a coupling between the counterpropagating states on the two edges of the trench. The simple single boson tunneling term obtained by approximating $\Psi(x) \approx \Psi_0$ does not couple the edges and leaves a state topologically equivalent to the gapless chiral edge on the trench. To solve this problem, we consider fluctuations in the BEC field as $\Psi(x) = \Psi_0 + \delta\Psi(x)$, where $\delta\Psi(x)$ is the bosonic fluctuation field in a three dimensional BEC. $\delta\Psi(x)$ can be integrated out to obtain an effective self-energy for the edge induced by the BEC, $\Sigma = \Sigma^{(1)} + \Sigma^{(2)}$, where $\Sigma^{(1)} = -\Psi_0\Delta\int_\Gamma d\xi e^{im\varphi(\xi)} + \mathrm{H.c.}$ is linear in Δ, while $\Sigma^{(2)}$ couples pairs of points directly across the trench, separated by a distance comparable to the trench's width W, as shown in Fig. 2. It is convenient to define the coordinate $x \in [0, L]$ along the trench of length L, together with the left- and right-moving fields $\varphi_L(x) = \varphi(x)$ and $\varphi_R(x) = \varphi(2L - x)$ propagating along the top and bottom edge, respectively. $\Sigma^{(2)}$ induces an interaction between the opposite edges of the BEC trench both via pairing as well as backscattering of bosons. On the other hand, we can use the freedom in tuning the trap potential to change the chemical potential in the BEC, which, in turn, changes the momentum of quasiparticles on the edge. A uniform change of density by ρ_0 implies $\varphi(\xi) \to \varphi(\xi) + 2\pi\rho_0\xi$, and thus gives rise to an oscillating phase in Eq. (3).

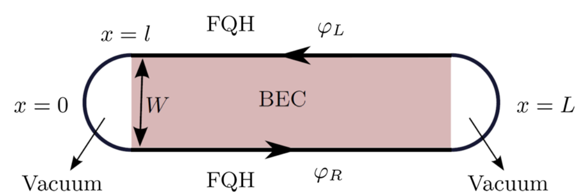

FIG. 2 (color online). The top view of a BEC trench within a FQH state. The trapping potential is engineered such that the boson density is uniform across the shaded area but vanishes in a small region of size l near the end points. Quantum fluctuations of the BEC couple opposite edges in the shaded region.

This phase would ensure momentum conservation [48], as the result of which $\Sigma^{(1)}$ is suppressed, and only those terms that conserve the total momentum contribute significantly to $\Sigma^{(2)}$. Physically, the only terms that satisfy these constraints are scattering terms on the same edge or pairing between opposite edges. The former simply renormalizes the velocity, while the latter is analogous to superconducting pairing. Taking the above constraints into account, the dominant, and local, contribution to the effective interaction is

$$V_{\rm eff} = -\lambda \int_l^{L-l} dx \cos(2m\phi), \tag{4}$$

where we have defined the (nonchiral) fields $\phi(x)$ and $\theta(x)$ by $\varphi_{L(R)}(x) = \phi(x) \mp \theta(x)$, and the coupling coefficient $\lambda \propto \Delta^2$. The new nonchiral fields are self-commuting at all points, but satisfy $[\phi(x), \theta(x')] = i(\pi/m)\Theta(x - x')$. For details of the derivation of the effective Hamiltonian, see the Supplemental Material [49].

The term $\cos(2m\phi)$ represents the effective pairing between the edges and is reminiscent of the sine-Gordon model [50]. The resulting Hamiltonian that describes the BEC trench (Fig. 2) can be written as

$$H_{\rm eff} \approx \frac{mv}{2\pi} \int_0^L dx[(\partial_x\phi)^2 + (\partial_x\theta)^2] - \lambda \int_l^{L-l} dx \cos(2m\phi), \tag{5}$$

where the continuity of $\partial_x\varphi(x)$ near $x \sim 0, L$ transforms into the boundary condition $\partial_x\phi(x = 0, L) = 0$. We note that, in addition, we are required to preserve the boundary conditions for θ at the end points: $\theta(x = L) = 0$ follows from the definition, while $\theta(x = 0) = \pi N/m$, where N is the total number of quasiparticles. The latter is due to the fact that the total density of bosons on both edges is $\rho = -\partial_x\theta/\pi$. However, similar to other restrictions on the Hilbert space, we will restore these boundary conditions at the end of the calculation. At large $\lambda > 0$, the sine-Gordon model [Eq. (5)] supports several ground states $|p\rangle$ characterized by the expectation values

$$\langle p|e^{i\phi(x)}|p\rangle \approx e^{i\pi p/m}, \qquad p = 0, 1, \ldots, 2m - 1, \tag{6}$$

for x away from the edges and in the limit of large L [50].

To restore the appropriate boundary conditions for $\theta(x{=}0,L)$, we notice that $\theta(x = L)$ commutes with $\phi(x)$ and $H_{\rm eff}$, hence it can be set to any value [$\theta(x = L) = 0$ in our case] without consequence for the single trench. On the other hand, $\theta(x = 0) = \pi N/m$ obeys a nontrivial commutation relation with $\phi(x)$ such that $[N, \phi] = -i$; however, $R = e^{i(\pi/m)N}$ commutes with $H_{\rm eff}$. The operator R, despite being a symmetry of the Hamiltonian, transforms $\phi(x)$ in a nontrivial way as

$$R\phi(x)R^\dagger = \phi(x) + \frac{\pi}{m}. \tag{7}$$

It then follows that, including the boundary conditions, different values of ϕ in Eq. (6) indeed correspond to the same energy. Exploiting the above symmetry, we can describe the ground states in a basis that also makes the operator R diagonal as

$$|n\rangle = \sum_{p=0}^{2m-1} e^{i\pi np/m}|p\rangle, \tag{8}$$

which satisfy $R|n\rangle = e^{i\pi n/m}|n\rangle$. Physically, n corresponds to the number of quasiparticles modulo pairs of bosons on the edge of a given trench. In our model thus far, this is a well-defined quantum number since the cosine term in Eq. (5) transfers only pairs of bosons from the condensate. Note that N takes $2m$ distinct eigenvalues. This allows us to associate a degeneracy of $\sqrt{2m}$ with each of the end points of the BEC trench (the "defects") in the limit of a large number of trenches where restrictions on the Hilbert space may be safely ignored. We will discuss Hilbert space constraints for a small number of trenches in the next section. Finally, we remark that the fractional part of the boson number on a trench, which is invariant under the addition of single bosons, is $Q = N/m \bmod 1$, while $Q_b \equiv [N/m - Q] \in \{0, 1\}$ defines the boson parity. The $\sqrt{2m}$ degeneracy thus also requires protection by a Z_2 symmetry due to boson parity.

Degeneracy.—So far we have focused on the spectrum of a single trench without the physical constraints of the Hilbert space. With a finite number of trenches, the total fractional charge Q should be 0 mod 1 [51]. Considering the double trench model of Eq. (2), the fractional boson numbers on the two trenches satisfy $Q_1 = -Q_2 = 0, 1/m, \ldots, 1 - 1/m$, while their boson parities $Q_{b,1} = 0, 1$ and $Q_{b,2} = 0, 1$ are unconstrained, which yields a total degeneracy of $D = 4m$ for two trenches. More generally, a system of k trenches with boson-parity conservation intact has the degeneracy

$$D = 2(2m)^{k-1}. \tag{9}$$

We shall discuss the (topological) robustness of this degeneracy, which partially survives the Z_2 symmetry breaking, after we explicitly construct operators that span the $4m$ degeneracy present in the two trench case.

Parafermion operators.—We now construct the operators spanning the above two-trench degeneracy. As remarked earlier, this degeneracy is spanned by the exchange of quasiparticles between the end points of the trenches. Let $U_l^\dagger$ ($U_r^\dagger$) be the exchange operator between the left (right) end points of the two trenches. They can be expressed as

$$U_{l,r}^\dagger \equiv T_{l,r}^\dagger B_{l,r}, \tag{10}$$

where $T^\dagger$ and $B^\dagger$ represent quasiparticle vertex operators acting on the top and bottom trench, respectively, and projected onto the ground-state sector. For example, $T_{l,r}^\dagger = e^{i\varphi_1(x=0,L)}$ adds a quasiparticle on one of the two ends of the first trench with the projection onto the ground state implicit. The operator B on the second trench can be defined similarly by $\varphi_1 \to \varphi_2$; however, we must insure that the exchange of quasiparticles between the two trenches respects the exchange statistics. This can be done consistently by including a Klein factor as $B^\dagger_{l,r} = e^{i\pi N_1/m} e^{i\varphi_2(x=0,L)}$, where N_1 is the number of quasiparticles on the first trench. With this construction, we now focus on the operator T defined above. Note that the chiral field at the ends is given by $\varphi_1 = \phi_1 - \theta_1$. As we discussed earlier, $\theta_1(L) = 0$ and $\theta_1(0) = \pi N_{\theta,1}/m \mod 2\pi$ with $N_{\theta,1} \equiv m(Q_{b,1} + Q_1)$ the total number of quasiparticles modulo $2m$ on the first trench. Since we are interested in the ground state sector, the field ϕ is roughly assumed to be pinned according to Eq. (6) over the entire edge as $\phi_1(0) \approx \phi_1(L) \approx 2\pi N_{\phi,1}/m$ with integer $N_{\phi,1}$. We find

$$T^\dagger_l = e^{i(\pi/m)(N_{\phi,1} - N_{\theta,1})}, \qquad T^\dagger_r = e^{i(\pi/m)N_{\phi,1}}, \tag{11}$$

which satisfy the algebra

$$(T^\dagger_{l,r})^{2m} = 1, \qquad T^\dagger_l T^\dagger_r = e^{-i(\pi/m)} T^\dagger_r T^\dagger_l. \tag{12}$$

These relations describe parafermions, a generalization of the fermionic algebra [23–26,52], see also Ref. [53].

The operators $B_{l,r}$ can be defined similar to Eq. (11) by including the Klein factor $\exp(i\pi N_{\theta,1}/m)$ explained above, and with $N_{\theta,1} \to N_{\theta,2} = m(-Q_1 + Q_{b,1})$ and $N_{\phi,1} \to N_{\phi,2}$. The parafermionic algebra implies that $(U^\dagger)^{2m} = 1$, which yields $2m$ degenerate ground states in a sector with a fixed total parity [the operator U does not change the total parity according to Eq. (10)]. With the twofold degeneracy due to the total parity, one recovers the full $4m$ degeneracy of the system.

Robustness.—Heretofore, we have focused on the Hamiltonian in Eq. (5). In principle, however, a single boson can tunnel to or from the BEC, which might arise from terms of the type $V = \cos[m\phi(x = 0, L)]$ near the end points of the trench. In fact, introducing the perturbation V within first order degenerate perturbation theory reduces the $4m$ fold degeneracy to $D' = m/2$; see the Supplemental Material [49] for details. While such a term clearly breaks the boson parity symmetry [hence $(Q_{b,1}, Q_{b,2})$ are no longer good quantum numbers], it also breaks a twofold degeneracy of the fractional number of quasiparticles Q_1. The remaining $m/2$-fold degeneracy is topologically protected, and we find in general that the degeneracy for k trenches is given by

$$D' = (m/2)^{k-1}. \tag{13}$$

This gives a quantum dimension of $\sqrt{m/2}$ for the defects ending the BEC trenches. However, if the vacuum regions (Fig. 2) are sufficiently large, single-boson tunneling is suppressed, and one recovers the degeneracy in Eq. (9).

We also briefly remark that the long-range fluctuations of the BEC can be considered effectively as long-range tunneling, and can provide another mechanism to break the boson-parity symmetry, while the quantum dimension in Eq. (13) will not be affected.

Preparation and detection.—There is some evidence that a $\nu = 1/2$ fractional Chern insulator has a continuous transition to a BEC, which allows for quasiadiabatic preparation of the former [54]. A similar procedure may exist for FQH states. The parafermions in the FQH state can be prepared by starting with a small island which can be grown to a trench in linear time by modulating the laser beams. Furthermore, Bragg spectroscopy can provide direct information about the topological phase of the system. For example, one should observe a zero bias peak at the end points of the trench [55]. Parafermionic zero modes can also be probed using braiding, which corresponds to the topologically protected manipulation of the underlying quantum information and which requires dynamically changing the geometry of the system, bringing different sets of parafermionic edge modes in close proximity to each other [23,24]. Such dynamical changes can easily be achieved by dynamically changing the laser beam used to create the BEC trenches.

It is worth pointing out that a theoretically simpler but experimentally more challenging approach would proceed by analogy with Ref. [56]: A BEC of diatomic molecules can be coherently dissociated into pairs of atoms, which readily gives the effective Hamiltonian in Eq. (4). We also mention that ultracold fermionic systems too can be utilized to create parafermionic zero modes; however, the pairing term has to be generated from a BEC of molecules of fermionic atoms, or some other paired state, which is, nevertheless, more complicated experimentally than a BEC trench proposed here. Furthermore, ultracold bosons are more common experimentally than ultracold fermions in part because they are easier to cool. Finally, for a fermionic system, one can use the solid-state proposals almost directly, while our bosonic model gives rise to qualitatively different results.

Conclusion.—In this work, we have considered a BEC trench in a FQH liquid, and showed that, in a certain regime, the combined system is in a topological phase, which is identified by the zero mode operators at the end points of the trench. These zero modes are shown to be parafermions, a generalization of the usual fermionic or bosonic algebra. We have also derived the topological degeneracy of the parafermionic modes, and examined their robustness against local perturbations. While we have focused on bosonic FQH states in the series $\nu = 1/m$, with m an even integer, our results may be extended in a straightforward manner to other bosonic and fermionic quantum Hall states in an ultracold-atomic setting [57].

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