# Supplementary Material:
# Monitoring-induced Entanglement Entropy and Sampling Complexity

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The supplementary information is organized as follows. In Sec. I we work out in detail the invariance of a master equation under a unitary mixing of the jump operators and how it affects the unravelings. In Sec. II we derive how the probability of a Markovian sequence of clicks can be obtained in the quantum jump picture. We derive a universal bound in Sec. III on the initial growth of entanglement entropy when $U$ is sampled from the Haar measure and finally, in Sec. IV, we illustrate that inducing entanglement in the trajectory states may reduce the statistical fluctuations for sampling a mixed-state density matrix.

## I. THE UNITARY MIXING OF JUMP OPERATORS: MASTER EQUATION VS. QUANTUM TRAJECTORIES

In this section, we illustrate in more detail the invariance of a master equation for linearly mixing jumps with a unitary, Eq. (2) from main text, and how this invariance breaks down in the trajectory picture.

The master equation of a dissipative system is given by

$$\partial_t \rho = -i[H,\rho] + \sum_j \Big( c_j \rho c_j^\dagger - \frac{1}{2}\{c_j^\dagger c_j, \rho\}\Big), \tag{S1}$$

and the trajectories for a given unraveling obey the stochastic equation [1, 2]

$$d\rho_\xi = -i[H,\rho_\xi]dt + \sum_j \Big( \langle c_j^\dagger c_j \rangle \rho_\xi - \frac{1}{2}\{c_j^\dagger c_j, \rho_\xi\}\Big)dt + \sum_j \left( \frac{c_j \rho_\xi c_j^\dagger}{\langle c_j^\dagger c_j\rangle} - \rho_\xi \right) dN_j. \tag{S2}$$

Here, $\xi$ labels a given realization of the stochastic variables $dN_j$, and $\rho_\xi = |\psi\rangle\langle\psi|$ is the stochastic pure state of the system in each realization. In the jump picture, it holds that $dN_j = 1$ with probability $\langle c_j^\dagger c_j\rangle dt$ and zero otherwise, so that $dN_j^2 = dN_j$ [1, 2]. Using that $\rho = \overline{\rho_\xi}$ and that $\overline{dN_j} = \langle c_j^\dagger c_j \rangle dt$, where $\overline{f}$ denotes the average of $f$ over the stochastic noise terms, one immediately recovers the Lindlbdlad master equation.

A direct calculation shows that all terms in the Lindblad master equation are invariant under $c_j \to c_j' = \sum_k U_{jk} c_k$. Indeed, using $\sum_j U_{lj}^* U_{kj} = \delta_{lk}$, we find

$$\sum_j c_j' \rho c_j'^\dagger = \sum_j \sum_{kl} U_{jk} c_k \rho U_{jl}^* c_l^\dagger = \sum_{kl} \delta_{lk} c_k \rho c_l^\dagger = \sum_k c_k \rho c_k^\dagger, \tag{S3}$$

and similarly for the other term $\propto \sum_j \{c_j^\dagger c_j, \rho\}$.

Importantly, this invariance does not hold generally for Eq. (S2) due to the last term, and this is precisely what motivates us to consider nonlinear trajectory-state quantities, such as the entanglement entropy. On the other hand, the probability of observing a quantum jump in time interval $dt$, $p_{\text{jump}}(t) = \sum_i \langle c_i^\dagger c_i \rangle dt$, is left invariant under the unitary transformation, which is why we discarded the explicit time dimension in the text and used the number of registered clicks instead. Recovering the explicit time dimension from an obtained jump-click trajectory is straightforwardly achieved by sampling the waiting times between clicks from the corresponding Poisson distributions.

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## II. PROBABILITY OF OBSERVING A JUMP SEQUENCE

To verify the equivalence with boson sampling, we need to evaluate the probability of observing a Markovian sequence of clicks $(m_1 \dots m_M)$, Eq. (4) in main text. Alternatively, the probability of such a sequence can be obtained by evaluating the temporal correlation function of the photonic state at the output ports of the LON $U$—see Ref. [3]. In this section, we derive in more detail how it naturally comes out of the quantum trajectory picture of the emitters.

Starting from the state $|\psi_0(M,N)\rangle = |\uparrow_M \downarrow_{N-M}\rangle$, when $k \leq M$ jumps have been detected in the sequence $(m_1, \dots, m_k)$, we know that the quantum state of the emitters is given by

$$|\psi\rangle_{m_1,\dots,m_k} = \mathcal{N}(m_1,\dots,m_k)\, c_{m_k} \dots c_{m_1} |\psi_0(M,N)\rangle, \tag{S4}$$

with the norm

$$\left|\mathcal{N}(m_1,\dots,m_k)\right|^2 = 1/\langle\psi_0(M,N)|c^\dagger_{m_1}\dots c^\dagger_{m_k} c_{m_k}\dots c_{m_1}|\psi_0(M,N)\rangle. \tag{S5}$$

The probability of sampling $c_{k+1}$ as the next jump, conditioned upon having observed the previous sequence $(m_1, \dots, m_k)$, is then obtained as the conditional probability

$$\begin{aligned} P\big(m_{k+1}|m_1,\dots,m_k\big) &= \frac{\langle\psi|c^\dagger_{k+1}c_{k+1}|\psi\rangle_{m_1,\dots,m_k}}{\sum_i \langle\psi|c^\dagger_i c_i|\psi\rangle_{m_1,\dots,m_k}} \\ &= \frac{\left|\mathcal{N}(m_1,\dots,m_k)\right|^2}{M-k}\,\langle\psi(M,N)|c^\dagger_{m_1}\dots c^\dagger_{m_k}\big(c^\dagger_{k+1}c_{k+1}\big)c_{m_k}\dots c_{m_1}|\psi(M,N)\rangle, \end{aligned} \tag{S6}$$

where the last step follows from $\sum_i \langle\psi|c^\dagger_i c_i|\psi\rangle_{m_1,\dots,m_k} = \sum_i \langle\psi|\sigma^+_i\sigma^-_i|\psi\rangle_{m_1,\dots,m_k}$ with the unitary transformation given by Eq. (2) in main text. From this, we can evaluate the probability of observing a sequence as a product of conditional probabilities

$$P(m_1,\dots m_M) = P(m_1) \times P\big(m_2|m_1\big) \times \dots \times P\big(m_M|m_1,\dots,m_{M-1}\big). \tag{S7}$$

Using Eq. (S6) for the different conditional probabilities, we arrive at Eq. (4) from the main text.

## III. BOUND ON INITIAL ENTANGLEMENT GROWTH

In this section, we formulate a universal upper bound for the initial growth rate of entanglement entropy when $M = N$, i.e. when all emitters start in the excited state, and $U_{N\times N}$ is drawn from the Haar measure.

For this, we know that, after registering one jump $c_i = \sum_j U_{ij}\sigma^-_j$, the quantum state of emitters is given by

$$|\psi_1(N)\rangle = c_i|\psi_0(N)\rangle = U_{i1}|\downarrow\uparrow\uparrow\uparrow\dots\rangle + U_{i2}|\uparrow\downarrow\uparrow\uparrow\dots\rangle + U_{i3}|\uparrow\uparrow\downarrow\uparrow\dots\rangle + \dots. \tag{S8}$$

For this state, we can compute the reduced density matrix of a subsystem $\mathcal{A}$, composed of $l$ sites, by tracing out the environment $\mathcal{B}$ composed of $N-l$ sites,

$$\rho_\mathcal{A} = \mathrm{tr}_\mathcal{B}\big[|\psi_1(N)\rangle\langle\psi_1(N)|\big] = p|\psi_{\mathcal{A},0}\rangle\langle\psi_{\mathcal{A},0}| + (1-p)|\psi_{\mathcal{A},1}\rangle\langle\psi_{\mathcal{A},1}|. \tag{S9}$$

Here, $p = \sum_{j\in\mathcal{A}}\left|U_{ij}\right|^2$ is the probability of finding the de-excitation in subsystem $\mathcal{A}$. Furthermore, $|\psi_{\mathcal{A},0}\rangle = |\uparrow_{l\in\mathcal{A}}\rangle$ is the quantum state when the jump is detected in the environment, and $|\psi_{\mathcal{A},1}\rangle = 1/\sqrt{p}\sum_{j\in\mathcal{A}}U_{ij}|\downarrow_j\uparrow_{l\neq j}\rangle$ the state if the jump occurs in $\mathcal{A}$. Hence, the reduced density matrix $\rho_\mathcal{A}$ is composed of a statistical mixture of two pure and orthogonal quantum states with a classical probability $p$, for which the entanglement entropy equals $S(p) = -p\log p - (1-p)\log(1-p)$.

Since the Von Neumann entanglement entropy is a concave function, meaning that $S\big(\sum_j \lambda_j\rho_j\big) \geq \sum_j \lambda_j S\big(\rho_j\big)$ for some statistical ensemble of density matrices $\rho_j$ with probabilities $\lambda_j$ (see e.g. Ref. [4]), an upper bound can be found by evaluating the entanglement entropy of the averaged density matrix of an ensemble. If we average instances from the $N\times N$ Haar measure in (S9), we know that $\mathbb{E}_U\big[|U_{ij}|^2\big] = 1/N$ and therefore that $\mathbb{E}_U[p] = l/N$, with $\mathbb{E}_U[\cdot]$ denoting the average over the measure of $N\times N$ Haar unitaries. Using the averaged probability for the statistical mixture given

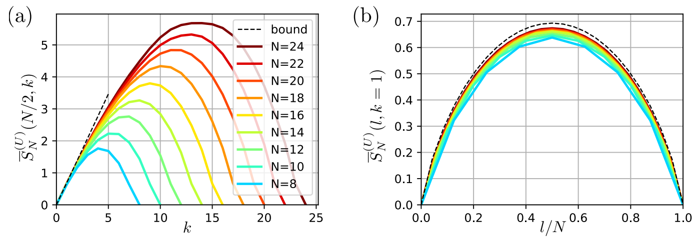


Figure S1. A comparison between the entanglement generated when $U$ is sampled from the $N \times N$ Haar measure; the analytical upper bound (S10) is almost saturated. (a) Evolution of half-chain (maximal) entanglement entropy after detection of $k$ clicks, for various $N$. (b) Entanglement subsystem profile after registering one jump click, same color codes as panel (a).

in (S9) after registering one click, a bound is found on the averaged entanglement entropy of a subsystem containing $l \leq N$ sites,

$$\overline{S}_N(l, k=1) \leq -\frac{l}{N} \log \frac{l}{N} - \frac{N-l}{N} \log \frac{N-l}{N}. \tag{S10}$$

This means that we find a universal bound for $x = l/N$ as $\overline{S}(x, k=1) \leq h(x) \equiv -x \log x - (1-x) \log(1-x)$. Numerically, we find that the bound also holds later, so that $\overline{S}(x, k) \leq k h(x)$.

In Fig. S1, we illustrate how bound (S10) is approached. The initial growth for $\overline{S}(N/2, k)$ lies close to the bound, as shown in Fig. S1(a). In Fig. S1(b), we illustrate that the bound from (S10) for the bipartite entanglement $\overline{S}(x, k=1)$ is approached when $N$ is increased.

Note also that the top of the curve $\overline{S}_{\max}$ seems to be slightly flattened for $N = 24$. The maximal bond dimension, set to $\chi_{\max} = 700$ for the MPS simulation, was not sufficient to capture all statistical fluctuations of entanglement entropy. Therefore the data for $N = 24$ was left out for $S_{\max}$ in Fig. 1b in the main text. We checked different sample trajectories to ensure that all data points $N \leq 22$ were not suffering from this issue.

## IV. SCALING OF THE AVERAGE ENTROPY AND STATISTICS OF THE UNRAVELING

We mention in the main text, just before conclusions and outlook, that inducing entanglement in the stochastic trajectories reduces the statistical trajectory fluctuations for sampling the averaged density matrix. In this section, we explain this in more detail for the case of $N$ emitters from the main text.

Let $\overline{\rho}_l = \sum_j \lambda_j \operatorname{tr}_{N-l}\left[|\psi^j\rangle\langle\psi^j|\right]$ denote the trajectory-averaged state of a subsystem of $l$ sites, where $\lambda_j$ is the probability with which state $|\psi^j\rangle$ occurs in the ensemble of trajectories, with $\sum_j \lambda_j = 1$. The trajectory-averaged von Neumann entropy of the subsystem, $\overline{S} = \sum_j \lambda_j S\left(\operatorname{tr}_{N-l}\left[|\psi^j\rangle\langle\psi^j|\right]\right)$, satisfies

$$\overline{S} \leq S(\overline{\rho}_l) \leq \overline{S} + \mathcal{H}(\lambda), \tag{S11}$$

where $\mathcal{H}(\lambda) = -\sum_j \lambda_j \ln \lambda_j$ is the (classical) Shannon entropy of the distribution $\{\lambda_j\}$ that characterizes the mixture [5, 6].

In the main text, we numerically studied $\overline{S}$. The entropy of the trajectory-averaged state is easy to obtain from the fact that, at the level of the master equation, each emitter remains in the excited state with a probability $p(t) = e^{-\gamma t}$. The entropy of $l$ emitters is then

$$S(\overline{\rho}_l; t) = l \times \Big(p(t) \log p(t) + (1 - p(t)) \log(1 - p(t))\Big). \tag{S12}$$

This entropy satisfies a volume law, $S(\overline{\rho}_l) \propto l$.

Using Eq. (S11), we find that, for the classical Shannon entropy of the mixture,

$$\mathcal{H}(\lambda) \geq S(\overline{\rho}_l) - \overline{S} \sim \mathcal{O}(l) - \overline{S}. \tag{S13}$$

This implies that whenever we find that $\overline{S}$ follows an area law, or scales slower than $S(\overline{\rho}_l)$ with $l$, the classical entropy $\mathcal{H}(\lambda)$ characterizing the unravelling, must compensate for this and scale with the volume of the system, $\mathcal{H}(\lambda) \sim \mathcal{O}(l)$. For numerical purposes, the number of distinct area-law trajectories in an unraveling needed to sample a master equation leading to a volume-law density matrix should scale exponentially to satisfy bound (S13). On the other hand, using volume-law trajectories, one might reach sufficient statistical accuracy after obtaining a set of samples with polynomial (or even constant) size. We plan to investigate this issue further in a follow-up work, with the goal of discovering optimal unravelings that have a balance between quantum entanglement (hardness of classically computing a given trajectory) and the number of samples needed (hardness of classical sampling) to acquire sufficient statistical accuracy.

[1] K. Jacobs and D. A. Steck, Contemp. Phys. **47**, 279–303 (2006).
[2] H. M. Wiseman and G. J. Milburn, *Quantum measurement and control* (Cambridge university press, 2009).
[3] V. Tamma and S. Laibacher, Quant. Inf. Proc. **15**, 1241 (2016).
[4] D. P. DiVincenzo, C. A. Fuchs, H. Mabuchi, J. A. Smolin, A. Thapliyal, and A. Uhlmann, in *NASA International Conference on Quantum Computing and Quantum Communications* (Springer, 1998) pp. 247–257.
[5] J. Dalibard, Y. Castin, and K. Mølmer, Phys. Rev. Lett. **68**, 580 (1992).
[6] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information: 10th Anniversary Edition*, 10th ed. (Cambridge University Press, USA, 2011).