



Enhancement of superconductivity with external phonon squeezing

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Squeezing of phonons due to the nonlinear coupling to electrons is a way to enhance superconductivity as theoretically studied in a recent work [Kennes *et al.*, *Nat. Phys.* **13**, 479 (2017)]. We study quadratic electron-phonon interaction in the presence of phonon pumping and an additional external squeezing. Interference between these two driving sources induces a phase-sensitive enhancement of electron-electron attraction, which we find as a generic mechanism to enhance any boson-mediated interactions. The strongest enhancement of superconductivity is shown to be on the boundary with the dynamical lattice instabilities caused by driving. We propose several experimental platforms to realize our scheme.

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Optical excitation of infrared-active (IR-active) phonon modes allows for ultrafast pumping of solid-state systems into nonequilibrium states exhibiting a wide range of exotic properties [1]. In particular, this enables manipulation of magnetic states [2], charge orders [3], and superconductivity [4,5]. A possible explanation of the transient enhancement observed in [4] is the parametric driving of Raman phonons by IR phonons [6,7]. Enhanced [8] quantum fluctuations of phonon modes lead to a stronger phonon-mediated attraction between electrons, thus enhancing the superconductivity.

Another proposed mechanism of transient superconductivity enhancement is based on a nonlinear coupling between electrons and phonons [9,10]. In this case, the squeezing of phonons is generated directly by the electron-phonon interaction, since the coupling is quadratic in the phonon operator. As shown in [10] due to this nonlinearity, the effective interaction is enhanced proportionally to the coherent excitation rate of the phonon mode.

In this Letter, we clarify the role of squeezing and study the possibility of enhancement of superconductivity by an additional external parametric drive. We consider a model that combines both ingredients—the linear and parametric driving of phonons that are nonlinearly coupled to electrons. In order to illustrate the influence of parametric driving on a bosonic degree of freedom [$a, a^\dagger = 1$] and introduce a related terminology, we define the Hamiltonian of a parametrically driven harmonic oscillator [11] with a bare frequency ω_0 as $H_{\text{PO}} = \omega_0 \hat{a}^\dagger \hat{a} - D(\hat{a}^2 e^{2i\omega_p t} + \hat{a}^{\dagger 2} e^{-2i\omega_p t})/2$ where D stands for the squeezing strength and ω_p is the parametric driving frequency. One can show that the expectation value of the quadrature $\hat{X}_\theta \equiv \hat{a} e^{i\theta + i\omega_p t} + \text{H.c.}$ decreases for $\theta = \pi$ and increases for $\theta = 0$, with respect to a local oscillator. We refer to these quadratures as “squeezed” and “antisqueezed,” respectively. Similarly, the corresponding retarded correlation function of these quadratures, which determines the strength of mediated interaction by these bosonic modes, can decrease or increase in a phase-sensitive fashion (see the Supple-

mental Material [12] for spins as an illustrative example). Consequently, the squeezing of proper phonon quadrature can significantly amplify the phonon-mediated electron-electron interaction. Moreover, the parametric drive can soften the phonon modes that further amplifies the electronic interaction. In this work, we demonstrate that such amplifications lead to the enhancement of superconductivity, by analytically and numerically employing the Migdal-Eliashberg theory. The superconducting critical temperature T_c is shown in Fig. 1(c) as a function of external phonon squeezing rate D . The external parametric drive that is the crucial element of our proposal can be achieved by either exploiting intrinsic photon-phonon coupling nonlinearities [13] or by using a parametric optical amplifier in an optical cavity to produce squeezed light [14] as schematically shown in Fig. 1(a).

To be specific, we study a two-dimensional superconductor interacting with an infrared-active optical phonon mode and consider the coupling to be quadratic in a phonon operator [4,9,10]. Linear coupling terms can also be present without affecting the results below. The full Hamiltonian of the system reads as $\hat{H}_{\text{full}} = \hat{H}_p + \hat{H}_e + \hat{H}_{e-p} + \hat{V}(t)$:

$$\begin{aligned} \hat{H}_p = & \underbrace{\sum_{\mathbf{q}} \omega_{\mathbf{q}} \hat{a}_{\mathbf{q}}^\dagger \hat{a}_{\mathbf{q}}}_{H_p} + \underbrace{\sum_{\mathbf{k}, \sigma} (\epsilon_{\mathbf{k}} - \mu) \hat{c}_{\mathbf{k}, \sigma}^\dagger \hat{c}_{\mathbf{k}, \sigma}}_{H_e} \\ & + \underbrace{\frac{g}{N} \sum_{\sigma, \mathbf{k}, \mathbf{q}, \mathbf{q}'} c_{\mathbf{k}+\mathbf{q}-\mathbf{q}', \sigma}^\dagger \hat{c}_{\mathbf{k}, \sigma} \hat{\phi}_{\mathbf{q}} \hat{\phi}_{-\mathbf{q}'}}_{H_{e-p}}, \end{aligned} \quad (1)$$

where N denotes the total number of lattice sites, \mathbf{q} is the lattice quasimomentum vector, $\omega_{\mathbf{q}}$ and $\epsilon_{\mathbf{k}}$ respectively stand for the phonon and electron dispersions, $\hat{c}_{\mathbf{k}, \sigma}$ is the electron annihilation operator, and $\hat{\phi}_{\mathbf{q}} \equiv \hat{a}_{\mathbf{q}} + \hat{a}_{-\mathbf{q}}^\dagger$ is the phonon displacement field operator. Phonons are linearly and parametrically driven at frequency ω_p with the corresponding driving strengths α and $D_{\mathbf{q}}$. The external driving Hamiltonian

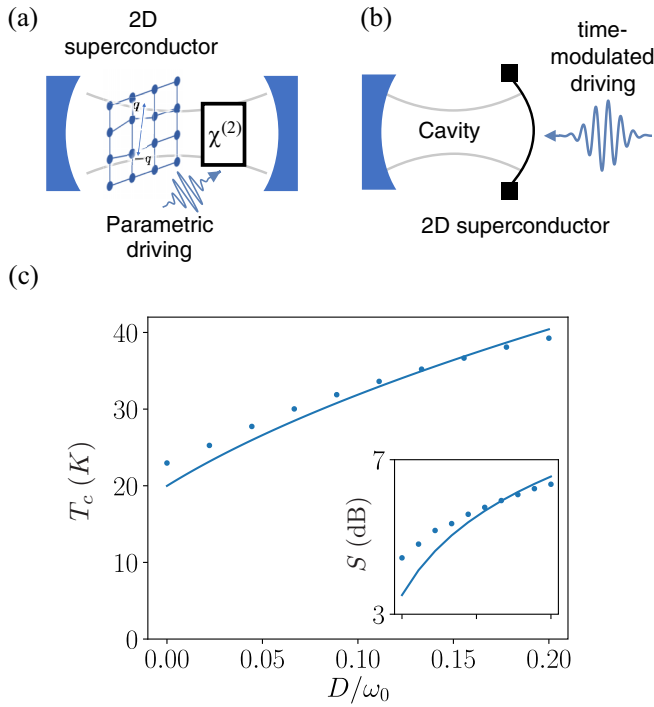


FIG. 1. Steady-state enhancement of superconducting transition temperature T_c . (a), (b) Sketch of the setup: 2D superconductor on an optomechanical membrane. Phonon squeezing is induced by (a) the squeezed light produced by the optical parametric oscillator and (b) time modulation of the classical input light. (c) Superconducting transition temperature as a function of the parametric driving strength D for optimal value of the detuning δ , assuming the phonon driving is fixed at $\alpha = 0.2\omega_0$: parametrically driven phonons (blue). The corresponding phonon excess noise $S \equiv N^{-1} \sum_{\mathbf{q}} \langle \hat{\phi}_{\mathbf{q}} \hat{\phi}_{-\mathbf{q}} \rangle$ is shown in the inset. The frequency of the IR-active phonon mode is taken to be $\omega_0 = 0.17$ eV corresponding to the frequency of the infrared-active phonon mode of the K_3C_{60} fulleride superconductor [6,9].

reads

$$\hat{V}(t) = 2\alpha \cos(\omega_p t + \theta_\alpha) \hat{\phi}_0 + \sum_{\mathbf{q}} D_{\mathbf{q}} \cos(2\omega_p t + \theta_D) \hat{\phi}_{\mathbf{q}} \hat{\phi}_{-\mathbf{q}}, \quad (2)$$

where $\theta_{\alpha,D}$ are the relative phases of linear and parametric drivings. As we discuss below, these phases allow one to control the strength of coupling to electrons. We note that the model becomes dynamically unstable at strong parametric drive $D_{\mathbf{q}}$ [11]. This is manifested in the exponential growth of the phonon displacement $\langle \hat{\phi}_{\mathbf{q}} \rangle$, as a function of time. Therefore, we impose $\omega_{\mathbf{q}} - \omega_p \geq D_{\mathbf{q}}$ to avoid such an instability.

The external drive induces a finite expectation value of the zero-momentum phonon mode $\langle \hat{\phi}_0 \rangle$. We treat this in terms of mean-field theory and keep quadratic fluctuations. We perform two unitary transformations of the Hamiltonian equation (1). First, we consider the frame, rotating at the phonon driving frequency ω_p , which transforms bosonic variables as $\hat{a}_{\mathbf{q}} \rightarrow \hat{a}_{\mathbf{q}} e^{-i\omega_p t}$. Second, we perform a shift of the zero-momentum bosonic variables $\hat{a}_{\mathbf{q}} \rightarrow \hat{a}_{\mathbf{q}} + \bar{a}_0 \delta_{\mathbf{q},0}$, where \bar{a}_0 denotes the adiabatic steady-state coherence to the lowest

order in $1/\omega_p$ (see Supplemental Material):

$$\bar{a}_0 \equiv \alpha \frac{D_0 e^{i(\theta_\alpha - \theta_D)} - \delta_0 e^{-i\theta_\alpha}}{\delta_0^2 - D_0^2}. \quad (3)$$

Finally, we perform the rotating-wave approximation and discard the rotating at frequencies $\pm 2\omega_p$, by assuming that the driving frequency ω_p is the largest energy scale in the system. As shown in the Supplemental Material, the effective coupling in the model is maximized for the following choice of driving phases $\theta_D = \pi$, $\theta_\alpha = 0$. This choice corresponds to antisqueezing of the quadrature to which electrons are coupled. With these approximations and neglecting all nonlinear and rotating contributions, the phonon Hamiltonian and the electron-phonon Hamiltonians are transformed as

$$\hat{H}_{\text{ph}} = \sum_{\mathbf{q}} \delta_{\mathbf{q}} \hat{a}_{\mathbf{q}}^\dagger \hat{a}_{\mathbf{q}} - \sum_{\mathbf{q}} \frac{D_{\mathbf{q}}}{2} (\hat{a}_{\mathbf{q}} \hat{a}_{-\mathbf{q}} + \hat{a}_{\mathbf{q}}^\dagger \hat{a}_{-\mathbf{q}}^\dagger), \quad (4)$$

$$\hat{H}_{\text{int}} = \frac{g_{\text{eff}}}{\sqrt{N}} \sum_{\sigma, \mathbf{q}} \hat{c}_{\mathbf{k}+\mathbf{q}, \sigma}^\dagger \hat{c}_{\mathbf{k}, \sigma} (\hat{a}_{\mathbf{q}} + \hat{a}_{-\mathbf{q}}^\dagger), \quad (5)$$

where the detuning $\delta_{\mathbf{q}} \equiv \omega_{\mathbf{q}} - \omega_p$ and the effective electron-phonon coupling is $g_{\text{eff}} = 2gl\bar{a}_0/\sqrt{N}$. Equation (4) is equivalent to a nondegenerate multimode parametric oscillator [11] below the parametric instability threshold for $\delta_{\mathbf{q}} \geq D_{\mathbf{q}}$, which can be diagonalized by means of the Bogolyubov transformation $\hat{a}_{\mathbf{q}} = \cosh(r_{\mathbf{q}}) \hat{b}_{\mathbf{q}} + \sinh(r_{\mathbf{q}}) \hat{b}_{-\mathbf{q}}^\dagger$ with $\zeta_{\mathbf{q}} = 2^{-1} \text{arctanh}(D_{\mathbf{q}}/\delta_{\mathbf{q}})$. We find $\hat{H}_{\text{ph}} = \sum_{\mathbf{q}} \sqrt{\delta_{\mathbf{q}}^2 - D_{\mathbf{q}}^2} \hat{b}_{\mathbf{q}}^\dagger \hat{b}_{\mathbf{q}}$ and $\hat{H}_{\text{int}} = N^{-1/2} g_{\text{eff}} \sum_{\sigma, \mathbf{q}} e^{i\zeta_{\mathbf{q}}} \hat{c}_{\mathbf{k}+\mathbf{q}, \sigma}^\dagger \hat{c}_{\mathbf{k}, \sigma} (\hat{b}_{\mathbf{q}} + \hat{b}_{-\mathbf{q}}^\dagger) e^{i\mathbf{q} \cdot \mathbf{r}_i}$. Close to the parametric instability $D_{\mathbf{q}} \sim \delta_{\mathbf{q}}$, the coupling scales as $e^{i\zeta_{\mathbf{q}}} \propto (1 - D_{\mathbf{q}}/\delta_{\mathbf{q}})^{-1/4}$. Equations (4) and (5) are therefore equivalent to a conventional Holstein model [15] with the softened phonons and an enhanced electron-phonon coupling. As we show below, the combination of these factors can lead to an enhanced T_c compared to the configuration without squeezing.

In order to show this enhancement, we consider the squeezed electron-phonon model of Eqs. (4) and (5) within the equilibrium Migdal-Eliashberg (ME) theory [16,17] and provide an estimate of the superconducting phase transition temperature T_c . ME theory relies on the Migdal theorem that allows one to neglect vertex corrections to the electron Green's function provided they are much faster than phonons. In the case of the effective Holstein model (4), (5), this is characterized by $\sqrt{\delta_{\mathbf{q}}^2 - D_{\mathbf{q}}^2} \ll E_F$, where E_F is the Fermi energy. The remaining equations for the electronic and phonon self-energies form a closed set of equations, which can be solved self-consistently, and we consider the formulation of the theory above the critical temperature $T \gtrsim T_c$.

We start by defining the fully renormalized imaginary-time propagators $\mathcal{G}_{\mathbf{k}}^{-1} = i\omega_n - (\epsilon_{\mathbf{k}} - \mu) - \Sigma_{\mathbf{k}}(i\omega_n)$, $\mathcal{D}_{\mathbf{k}}^{-1} = \mathcal{D}_{\mathbf{k}}^{(0)-1}(i\omega_m) - \Pi_{\mathbf{k}}(i\omega_n)$, where the unperturbed squeezed phonon propagator is $\mathcal{D}_{\mathbf{q}}^{(0)}(i\omega_m) = -2(\delta_{\mathbf{q}} + D_{\mathbf{q}})/(\omega_m^2 + \delta_{\mathbf{q}}^2 - D_{\mathbf{q}}^2)$ and $\omega_n = \pi(2n+1)/\beta$, $\omega_m = 2\pi m/\beta$, $m, n \in \mathbb{Z}$ denote fermionic and bosonic Matsubara frequencies, respectively.

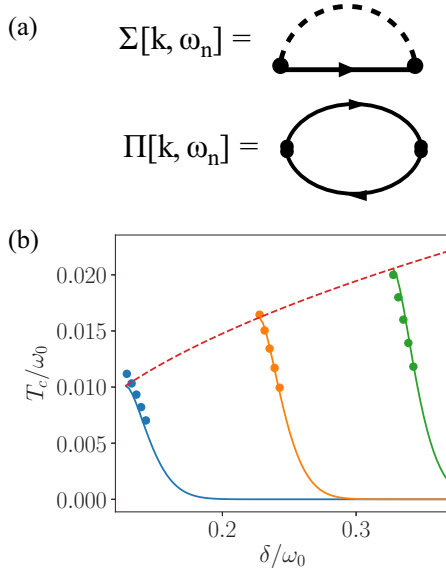


FIG. 2. Enhancement of superconductivity. (a) Electronic and phononic self-energies within Migdal-Eliashberg theory. Solid and dashed lines respectively stand for the fully renormalized electron and phonon propagators. (b) Superconducting T_c as a function of the detuning δ : configuration without squeezing $D = 0$ (blue), $D = 0.1\omega_0$ (orange), and $D = 0.2\omega_0$ (green); and optimal values (red dashed). Dotted curves correspond to the exact numerical solution of the Migdal-Eliashberg equations (6)–(8) on a discretized momentum-frequency lattice.

The electronic and phononic self-energies obey the equations [15], as diagrammatically shown in Fig. 2(a):

$$\Sigma_{\mathbf{k}}(i\omega_n) = -\frac{g_{\text{eff}}^2}{\beta N} \sum_{\mathbf{m}, \mathbf{q}} \mathcal{D}_{\mathbf{k}-\mathbf{q}}(i\omega_n - i\omega_m) \mathcal{G}_{\mathbf{q}}(i\omega_m), \quad (6)$$

$$\Pi_{\mathbf{k}}(i\omega_n) = \frac{2g_{\text{eff}}^2}{\beta N} \sum_{\mathbf{m}, \mathbf{q}} \mathcal{G}_{\mathbf{q}}(i\omega_m) \mathcal{G}_{\mathbf{q}-\mathbf{k}}(i\omega_m - i\omega_n). \quad (7)$$

These equations define the properties of the normal state of the electron gas. In order to find the superconducting transition temperature, we solve the linearized self-consistent equation for the pairing vertex $\Gamma_{\mathbf{k}}$:

$$\Gamma_{\mathbf{k}}(i\omega_n) = -\frac{g_{\text{eff}}^2}{N\beta} \sum_{\mathbf{q}} \mathcal{D}_{\mathbf{k}-\mathbf{q}}(i\omega_n - i\omega_m) \Gamma_{\mathbf{q}}(i\omega_m) \times \mathcal{G}_{\mathbf{q}}(i\omega_m) \mathcal{G}_{-\mathbf{q}}(-i\omega_m). \quad (8)$$

The highest-temperature solution of this equation defines the critical temperature T_c . We provide an analytical solution of Eqs. (6)–(8) under several simplifying assumptions. In particular, we assume that the detuning $\delta_{\mathbf{q}}$ and the squeezing parameter $D_{\mathbf{q}}$ do not depend on momentum \mathbf{q} . In this case, the only momentum dependence in Eqs. (6) and (7) is due to the electron polarization operator. The latter Eq. (7) contains static $\omega_n = 0$ and dynamical contributions $\omega_n \neq 0$. The static contribution is responsible for the phonon softening due to the interaction with electrons and it is generally important at strong couplings. In addition, it effectively enhances the electron-phonon interaction [18]. The dynamical contribution

describes the Landau damping. We neglect the dynamical contribution as it is smaller than the first Matsubara frequency term in the denominator of \mathcal{D} in the relevant temperature ranges [18]. In addition, we restrict the polarization operator in Eq. (7) to its zeroth Matsubara component taken with respect to the unperturbed fermionic Green's function: $\Pi_{\mathbf{k}}(i\omega_n) \approx -2\nu_0 g_{\text{eff}}^2$, where $\nu_0 \equiv N^{-1} \sum_{\mathbf{k}} \delta(E_F - \epsilon_{\mathbf{k}})$ is the density of states at the Fermi energy E_F . We note that according to this definition, ν_0 has the dimension of inverse energy. Under these assumptions the renormalized phonon propagator takes the following form:

$$\mathcal{D}(i\omega_n) = \frac{-2(\delta + D)}{\omega_n^2 + (\delta^2 - D^2)(1 - 2\lambda_0)}, \quad (9)$$

where the effective electron-phonon coupling is defined as $\lambda_0 = 2\nu_0 g_{\text{eff}}^2 / (\delta - D)$. The antisqueezing manifests itself as an excess noise of the phonon field $\hat{\phi}_{\mathbf{q}}$, which is found by taking the Matsubara frequency sum in Eq. (9) in the $T \rightarrow 0$ limit $\langle \hat{\phi}_{\mathbf{q}} \hat{\phi}_{-\mathbf{q}} \rangle = -\beta^{-1} \sum_n \mathcal{D}_{\mathbf{q}}(i\omega_n) \approx \sqrt{(\delta + D)/[(1 - 2\lambda_0)(\delta - D)]}$. We see that the phonon fluctuations are enhanced by interaction with electrons and by external squeezing in a multiplicative way.

Since the right-hand sides of Eqs. (6) and (8) do not depend on momentum k , the dependence can be eliminated by taking an average over Fermi surface $\Gamma_n \rightarrow \langle \Gamma_{\mathbf{k}}(i\omega_n) \rangle_{\text{FS}}$, $\Sigma_n \rightarrow \langle \Sigma_{\mathbf{k}}(i\omega_n) \rangle_{\text{FS}}$. An approximate analytical solution of these equations is known [19,20], and yields the following expression for the critical temperature:

$$T_c = \frac{\sqrt{(\delta^2 - D^2)(1 - 2\lambda_0)}}{1.2} e^{-1.04[(\lambda_{\text{eff}} + 1)/\lambda_{\text{eff}}]}, \quad (10)$$

where the effective coupling strength is defined as $\lambda_{\text{eff}} = \lambda_0 / (1 - 2\lambda_0)$ [18], and the first term in this expression stands for the effective phonon bandwidth, which corresponds to the poles of Eq. (9) with respect to the Matsubara frequency. At strong coupling, λ_0 , the system undergoes a transition to charge-density phase [17,18,21]. In Eq. (10), it manifests itself as singularity of λ_{eff} at $\lambda_0^{\text{cr}} = 0.5$. Due to the vertex corrections neglected in Eqs. (6)–(8), the exact Monte Carlo treatment of the Holstein model [17] predicts a slightly different value $\lambda_0^{\text{cr}} \approx 0.4$. In the following, we will restrict all system parameters such that $\lambda_0 \leq \lambda_0^{\text{cr}}$ in order to avoid this instability.

We now analyze Eq. (10) by varying δ and D while assuming g_{eff} is fixed. In the absence of squeezing ($D = 0$), we find the maximum T_c with respect to the detuning δ being equal to $T_c^{\text{max}} \approx 0.4 g_{\text{eff}}^2 \nu_0$. This value, being expressed in terms of the optimal detuning, is equal to $T_c^{\text{max}} \approx 0.08\delta$, which reproduces the known result [17,22]. In order to study the influence of squeezing on the superconducting temperature, we assume the squeezing parameter D is fixed to some positive value. In this case a new maximum with respect to δ is straightforwardly found to be $T_c^{\text{max}}(D) \approx 0.25 \sqrt{g_{\text{eff}}^2 \nu_0 D}$ in the limit when $D \gg g_{\text{eff}}^2 \nu_0$. It is achieved at $\delta^{\text{max}} \approx D + 5g_{\text{eff}}^2 \nu_0$. This combination of squeezing and detuning saturates the bare electron-phonon coupling to $\lambda_0 \approx \lambda_0^{\text{cr}}$, which is approximately independent of D and δ . The effective phonon bandwidth for the optimal detuning scales as $\sqrt{(\delta^2 - D^2)(1 - 2\lambda_0)} \propto$

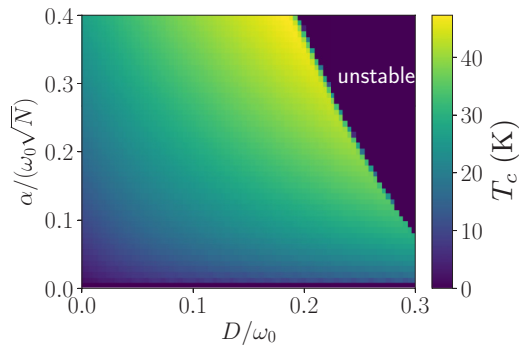


FIG. 3. Superconducting critical temperature T_c as a function of the linear driving α and parametric driving strength D . The detuning is chosen to produce the strongest effect at each point. The strongest enhancement is achieved at the boundary of the lattice instability region.

$\sqrt{g_{\text{eff}}^2 \nu_0 D}$, which determines the scaling of T_c at large squeezing. The enhancement can therefore be seen as increasing the effective bandwidth of phonons while keeping the effective coupling fixed to its maximal value [17,22]. In general, g_{eff} also depends on D and δ due to being proportional to the steady-state phonon occupation. The presented analysis can be straightforwardly extended to take this into account.

We now compare the analytical prediction for the critical temperature with the numerical self-consistent solution of Eqs. (6) and (8) performed in a discretized 52×52 momentum and 400 Matsubara frequency lattice space. We consider the external driving $\alpha = 0.25\omega_0$ to be fixed, while we vary the detuning δ . The critical temperature as a function of the detuning is shown in Fig. 2 for several values of the squeezing parameter D . The smallest detuning of all curves corresponds to $\lambda_0 \approx \lambda_0^{\text{cr}}$. The maximum T_c is achieved at the lowest possible δ in agreement with the analytical expression provided above. We study the effect of linear α and parametric D driving on superconducting T_c for optimal values of detuning δ (Fig. 3). The external squeezing allows one to achieve strong enhancement at much lower driving intensities α , and the strongest effect is achieved on the boundary of the lattice instability regimes.

We now discuss two possible experimental realizations of our idea to generate phonon squeezing. The first proposal exploits the intrinsic photon-phonon coupling nonlinearities [13]. For illustration purposes, we consider a simplified model of two-dimensional electron lattice gas with the nearest-neighbor tunneling rate $J \approx 0.2\omega_0$, corresponding to the K_3C_{60} fulleride superconductor [6,9]. We assume that the infrared-active phonon mode with Debye frequency $\omega_0 \approx 0.17$ eV being driven by a bichromatic light at frequencies ω_p and $2\omega_p$. For an estimate of achievable phonon parametric driving rate, we take the values achieved with phonon parametric amplification [13] $D \propto 0.1\omega_0$ as achieved at field values of the order of 10 MV/cm. Here we focus only on pairing induced by external driving. In our simulations we consider the electron-phonon coupling coefficient $g \approx 0.1\omega_0$ [9] and the electron density of states $\nu_0 \approx 0.6\omega_0^{-1}$. We note that in deriving Hamiltonians (4) and (5) we neglected

the terms rotating at $\propto 2\omega_0$ [23] (see Supplemental Material) which may induce heating for broader-band materials. Our simplified analysis can be extended with taking these rotating terms into account perturbatively [6]. With the parameters above we estimate the bare electron-phonon coupling $g^2 \nu_0 \approx 0.006\omega_0$. The corresponding values of the critical temperature are shown in Fig. 1(c).

In the second approach, we consider a two-dimensional [24] superconductor optomechanical membrane optomechanically coupled to a cavity mode as shown in Figs. 1(a) and 1(b). Experimentally, the cavity optomechanical coupling to the two-dimensional (2D) van der Waals system has recently been demonstrated in [25–27]. The main challenge in this case is to control the high-frequency phonons as the critical temperature, Eq. (10), is proportional to the overall frequency range of the phonon modes. Coupling of light to high-frequency phonons has been demonstrated in several setups including the optomechanical disk resonators [28] and high-frequency bulk acoustic phonons [29]. We consider two possible ways of phonon squeezing as illustrated in Fig. 1. First, squeezing can be achieved via hybridization with photons which are parametrically driven [14,30]. Alternatively, as we show in the Supplemental Material [31], squeezing of membrane can be performed by a very specific time modulation of incoming light. By assuming a frequency range of the order of 100 GHz, as achieved in resonators based on acoustic distributed Bragg reflectors [32–37], we can estimate the achievable enhancement to be of the order of $T_c \propto 3$ K for the same parameter ratio as provided in the previous paragraph. The main limiting factor is the effective phonon bandwidth which is substantially reduced due to squeezing close to the parametric instability. However, the analysis presented in this Letter is restricted to the isotropic case, i.e., when $D_{\mathbf{q}} = D$, $\delta_{\mathbf{q}} = \delta$. The momentum dependence of the $D_{\mathbf{q}}$ and $\delta_{\mathbf{q}}$, which is generally present in experiment, provides an additional degree of freedom. In particular, this allows one to control the effective phonon dispersion independently of the coupling strength. The enhancement of electron interaction may be expected in the case of parametrically driving only the phonons with $q = 2k_F$, where k_F is the Fermi momentum.

In conclusion, we studied the enhancement of superconductivity due to an externally induced squeezing. The phase-sensitive squeezing enhances quadrature fluctuations of the phonon field leading to exponentially stronger interaction, while reducing the spectral bandwidth of phonons. We study the competition of these two effects numerically and analytically and find a parameter range of enhanced superconductivity. The effective squeezed Holstein model describing the system allows one also to dynamically suppress coupling to a certain range of phonon modes. The strength of the suppression is exponential. This can be very useful in the case when superconductivity competes with other types of instabilities, e.g., charge-density wave instability. By decoupling from the phonon modes responsible for the instability, one can enhance the superconducting transition. This opens up a way to engineer an effective electron-phonon interacting model which suppresses polaronic/charge-density wave tendencies.

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Supplemental Material for: Enhancement of superconductivity with external phonon squeezing

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I. ENHANCEMENT IN ENSEMBLE OF TWO-LEVEL SYSTEMS

To illustrate the effect of squeezing on the mediated interaction and the phase-sensitive nature of the increase/decrease of the interaction strength, we study the case of spins coupled to a bosonic modes. While the focus of the main text is on electrons in 2D, here we present the key idea using a toy model with spins. Moreover, such a phase-sensitive nature was not discussed in previous works on the subject [13]. To be concrete, we study interaction of an ensemble of N two-level systems (TLS) with the squeezed bosonic mode. We demonstrate a phase-sensitive enhancement analogous to those studied in the main text. We first define the raising/lowering operators for TLS as σ_i^\pm and the bosonic annihilation operator as a . We consider the following Hamiltonian:

$$H = \omega_0 a^\dagger a + \omega_a \sum_{i=1}^N \sigma_i^+ \sigma_i^- + \frac{D}{2} (e^{2i\phi_D} a^2 e^{2i\omega_a t} + e^{-2i\phi_D} e^{-2i\omega_a t} a^{\dagger 2}) + g (a + a^\dagger) \sum_{i=1}^N (\sigma_i^+ + \sigma_i^-)$$

where ω_a is the transition energy of a two-level system, ω_0 is the frequency of a bosonic mode and D denotes the squeezing rate. We transform into rotating frame $a \rightarrow a e^{-i\omega_a t - i\phi_D}$ and $\sigma_i^- = \sigma_i^- e^{-i\omega_a t}$. By denoting the detuning $\delta = \omega_0 - \omega_a$ we get and performing the rotating-wave approximation (RWA):

$$H = \delta a^\dagger a + \frac{D}{2} (a^2 + a^{\dagger 2}) + g (e^{-i\phi_D} a J^+ + e^{i\phi_D} a^\dagger J^-),$$

where we denoted $J^\pm = \sum \sigma_i^\pm$, $[J^+, J^-] = 2J^z$. By performing the Bogolyubov transformation $a = ub + v\hat{b}^\dagger$ with $u = \cosh[r]$, $v = \sinh[r]$, $r = \operatorname{arctanh}[-D/\delta]$ we get:

$$H = \sqrt{\delta^2 - D^2} b^\dagger b + g(b\{e^{-i\phi_D} u \sum_i \sigma_i^+ + e^{i\phi_D} v \sum_i \sigma_i^-\} + \text{H.c.})$$

We now perform adiabatic elimination of bosonic mode by means of the Schrieffer-Wolff transformation:

$$H' = e^{-S} H e^S = H + \frac{1}{2} [H, S] + \dots \quad (1)$$

with

$$S = \frac{2g}{\sqrt{\delta^2 - D^2}} (b\{e^{-i\phi_D} u J^+ + e^{i\phi_D} v J^-\} - b^\dagger\{e^{-i\phi_D} v J^+ + e^{i\phi_D} u J^-\}),$$

By taking the necessary commutators in (1) we get:

$$H' = -g^2 \left\{ \frac{e^{-2i\phi_D} D}{\delta^2 - D^2} J^{+2} + \frac{D e^{2i\phi_D}}{\delta^2 - D^2} J^{-2} \right\} - g^2 \frac{\delta}{\delta^2 - D^2} (J^+ J^- + J^- J^+) - \frac{g^2}{\sqrt{\delta^2 - D^2}} (2b^\dagger b + 1) J^z$$

In order to represent this result in a more physically appealing form, we transform variables as $J^\pm e^{\mp i\phi_D} \rightarrow J^\pm$ and denoting $J^{x/y} = \frac{1}{2} (J^+ \pm iJ^-)$ we find:

$$H' = -g^2 \left\{ \frac{1}{\delta - D} (J^x)^2 + \frac{1}{\delta + D} (J^y)^2 \right\} - \frac{g^2}{\sqrt{\delta^2 - D^2}} (2b^\dagger b + 1) J^z$$

We therefore find that interaction is enhanced in one quadrature and decreased in the other. We note that the definition of $J^{x/y}$ is arbitrary without external reference.

II. DERIVATION OF THE EFFECTIVE HAMILTONIAN

In this section, we provide technical details of the derivation of the effective Hamiltonian Eqs. (4, 5) of the main text. We treat phonon dynamics in terms of mean-field theory and keep quadratic fluctuations. We represent phonon operators as $\hat{\phi}_{\mathbf{q}} = \langle \hat{\phi}_0 \rangle \delta_{\mathbf{q}, \mathbf{0}} + \tilde{\hat{\phi}}_{\mathbf{q}}$. The mean-field set of equations reads:

$$\frac{d}{dt} \langle \hat{\phi}_0 \rangle = \omega_0 \langle \hat{\pi}_0 \rangle, \quad (2)$$

$$\frac{d}{dt} \langle \hat{\pi}_0 \rangle = -\omega_0 \langle \hat{\phi}_0 \rangle - 4\alpha \cos(\omega_p t + \psi_\alpha) - 4\langle \hat{\phi}_0 \rangle D_0 \cos(2\omega_p t + \psi_D), \quad (3)$$

where we defined the ‘‘momentum’’ operator $\hat{\pi}_0 \equiv i(a_0^\dagger - a_0)$. In deriving these equations we neglected coupling to the electrons. The latter may induce frequency shift. The resulting effective Hamiltonian for fluctuating part is (we omit tildes for shortness):

$$H_{\text{ph}} = \sum_{\mathbf{q}} \omega_{\mathbf{q}} \hat{a}_{\mathbf{q}}^\dagger \hat{a}_{\mathbf{q}} + \sum_{\mathbf{q}} D_{\mathbf{q}} \cos(2\omega_p t + \psi_D) \hat{\phi}_{\mathbf{q}} \hat{\phi}_{-\mathbf{q}}, \quad (4)$$

$$H_{\text{int}} = \frac{2g}{N} \sum_{\sigma, k, q} \hat{c}_{\mathbf{k}+\mathbf{q}, \sigma}^\dagger \hat{c}_{\mathbf{k}, \sigma} \hat{\phi}_{\mathbf{q}} \langle \hat{\phi}_0 \rangle + \frac{g}{N} \sum_{\sigma, k, q} \hat{c}_{\mathbf{k}, \sigma}^\dagger \hat{c}_{\mathbf{k}, \sigma} \langle \hat{\phi}_0 \rangle^2, \quad (5)$$

where we neglected the quadratic in $\hat{\phi}$ electron-phonon coupling in H_{int} . The second term in Eq. (5) stands for the renormalization of the chemical potential. As shown in section below, the mean field value $\langle \hat{\phi}_0(t) \rangle \approx 2\text{Re}[\bar{a}_0 e^{-i\omega_p t}]$, where

$$\bar{a}_0 = \alpha \frac{D_0 e^{i(\psi_\alpha - \psi_D)} - \delta_0 e^{-i\psi_\alpha}}{(\delta_0^2 - D_0^2)}.$$

We now transform into the frame rotating at ω_p and neglect all high-frequency rotating terms:

$$H_{\text{ph}} = \sum_{\mathbf{q}} (\omega_{\mathbf{q}} - \omega_p) a_{\mathbf{q}}^\dagger \hat{a}_{\mathbf{q}} + \sum_{\mathbf{q}} \frac{D_{\mathbf{q}}}{2} \left(e^{i\psi_D} \hat{a}_{\mathbf{q}} a_{-\mathbf{q}} + e^{-i\psi_D} a_{-\mathbf{q}}^\dagger a_{\mathbf{q}}^\dagger \right) \quad (6)$$

$$H_{\text{int}} = \frac{2g}{N} \sum_{\sigma, k, q} c_{\mathbf{k}+\mathbf{q}, \sigma}^\dagger c_{\mathbf{k}, \sigma} \left(\hat{a}_{\mathbf{q}} \bar{a}_0^* + a_{-\mathbf{q}}^\dagger \bar{a}_0 \right). \quad (7)$$

We find an effective Holstein model provided in the main text. In deriving the Hamiltonian (6),(7) we neglected the following terms rotating at $2\omega_p$:

$$H^{\text{rot}} = \sum_{\mathbf{q}} \frac{D_{\mathbf{q}}}{2} \left(e^{i(2\omega_p t + \psi_D)} + e^{-i(2\omega_p t + \psi_D)} \right) \left(a_{\mathbf{q}} a_{\mathbf{q}}^\dagger + a_{-\mathbf{q}}^\dagger a_{-\mathbf{q}} \right), + \frac{2g}{N} \sum_{\sigma, k, q} \hat{c}_{\mathbf{k}+\mathbf{q}, \sigma}^\dagger \hat{c}_{\mathbf{k}, \sigma} \left(a_{\mathbf{q}} \bar{a}_0 e^{-2i\omega_p t} + a_{-\mathbf{q}}^\dagger \bar{a}_0^* e^{2i\omega_p t} \right) + \frac{g}{N} \sum_{\sigma, k, q} \hat{c}_{\mathbf{k}, \sigma}^\dagger \hat{c}_{\mathbf{k}, \sigma} \left(\bar{a}_0^2 e^{-2i\omega_p t} + \bar{a}_0^{*2} e^{2i\omega_p t} \right).$$

This approximation is valid as soon as there are no possible resonant transitions in the electron gas caused by the rotating terms.

III. STEADY STATE PHONON PROPERTIES

Here we consider properties steady state properties of phonon modes. We first approximately solve the mean

field set of equations Eqs. (2, 3). For that we assume $\langle \hat{\phi}_0(t) \rangle \approx 2\text{Re}[\bar{a}_0 e^{-i\omega_p t}]$ and find \bar{a}_0 neglecting coupling to higher frequency components. We note that by making this ansatz we neglect terms rotating at $3\omega_p, 5\omega_p, \dots$ which produce only rapidly rotating terms in Eqs. (2, 3).

This results in the following equations:

$$\begin{aligned} (\omega_0^2 - \omega_p^2) \bar{a}_0 &= -2D_0 \omega_0 \bar{a}_0^* e^{-i\psi_D} - 2\alpha \omega_0 e^{-i\psi_\alpha} \\ (\omega_0^2 - \omega_p^2) \bar{a}_0^* &= -2D_0 \omega_0 \bar{a}_0 e^{i\psi_D} - 2\alpha \omega_0 e^{i\psi_\alpha} \end{aligned}$$

The solution is:

$$\bar{a}_0 = \alpha \frac{4D_0 \omega_0^2 e^{i(\psi_\alpha - \psi_D)} - 2\omega_0 (\omega_0^2 - \omega_p^2) e^{-i\psi_\alpha}}{(\omega_0^2 - \omega_p^2)^2 - 4D_0^2 \omega_0^2} \quad (8)$$

Using $\omega_0 = \omega_p + \delta_0$ and expanding in the limit $\omega_p/\delta_0 \rightarrow \infty$ we find:

$$\bar{a}_0 \approx \alpha \frac{D_0 e^{i(\psi_\alpha - \psi_D)} - \delta_0 e^{-i\psi_\alpha}}{(\delta_0^2 - D_0^2)} \quad (9)$$

Phonon propagator In Eq. (7) electrons are effectively coupled to the phonon field $\hat{\Phi}_{\mathbf{q}} = \left(\hat{a}_{\mathbf{q}} e^{-i \arg(a_0)} + a_{-\mathbf{q}}^\dagger e^{i \arg(a_0)} \right)$. We now derive bare propagator of this field

$$\mathcal{D}^R[\omega] = -i \int_0^\infty e^{i\omega t} \langle [\Phi_{\mathbf{q}}(t), \Phi_{-\mathbf{q}}(0)] \rangle$$

We start with the set of Heisenberg equations of motion with respect to the bare phonon Hamiltonian Eq. (6):

$$\begin{aligned} \frac{d}{dt} \hat{a}_{\mathbf{q}} &= -i\delta_{\mathbf{q}} \hat{a}_{\mathbf{q}} - iD_{\mathbf{q}} e^{-i\psi_D} a_{-\mathbf{q}}^\dagger \\ \frac{d}{dt} a_{-\mathbf{q}}^\dagger &= i\delta_{\mathbf{q}} a_{-\mathbf{q}} + iD_{\mathbf{q}} e^{i\psi_D} \hat{a}_{\mathbf{q}} \end{aligned}$$

Solving them we find the imaginary-time propagator:

$$\mathcal{D}_{\mathbf{q}}[i\omega_n] = \frac{2(D_{\mathbf{q}} \cos(\psi_D + 2 \arg(a_0)) - \delta_{\mathbf{q}})}{\omega_n^2 + \delta^2 - D_{\mathbf{q}}^2} \quad (10)$$

Numerator of this expression is maximized by e.g. the following choice of driving phases $\psi_D = \pi$ and $\psi_\alpha = 0$:

$$\mathcal{D}_{\mathbf{q}}[i\omega_n] = \frac{-2(D_{\mathbf{q}} + \delta_{\mathbf{q}})}{\omega_n^2 + \delta_{\mathbf{q}}^2 - D_{\mathbf{q}}^2}$$

A. Phonon dissipation

We now consider the possible phonon dissipation. We note that Migdal-Eliashberg formalism takes into account

the dissipation due to the interaction with electrons. We therefore consider the dissipation that comes from other processes like phonon-phonon interactions [5]. We take dissipation in the simplest Markovian form with the momentum- and frequency- independent rate γ [5]. It is straightforward to show that the bare phonon propagator is modified as follows [6]:

$$\mathcal{D}_{\mathbf{q}}^{(0)}[i\omega_n] = \frac{-2(D_{\mathbf{q}} + \delta_{\mathbf{q}})}{\omega_n^2 + \gamma^2 + 2\gamma|\omega_n| + \delta_{\mathbf{q}}^2 - D_{\mathbf{q}}^2} \quad (11)$$

Renormalizing this propagator with the fermion loop as discussed in the main text we find:

$$\mathcal{D}_{\mathbf{q}}[i\omega_n] = \frac{-2(D_{\mathbf{q}} + \delta_{\mathbf{q}})}{\omega_n^2 + \gamma^2 + 2\gamma|\omega_n| + (\delta_{\mathbf{q}}^2 - D_{\mathbf{q}}^2)(1 - 2\lambda_0)}. \quad (12)$$

We note that it corresponds to the Lorentz spectral density of the propagator:

$$\mathcal{D}_{\mathbf{q}}[i\omega_n] = \frac{1}{\pi} \int_0^\infty dx^2 \frac{\{\rho(x) - \rho(-x)\}}{x^2 + \omega_n^2} \quad (13)$$

with

$$\rho(x) = \left(\frac{D_{\mathbf{q}} + \delta_{\mathbf{q}}}{\tilde{\delta}_{\mathbf{q}}} \right) \frac{\gamma}{\gamma^2 + (x - \tilde{\delta}_{\mathbf{q}})^2}, \quad (14)$$

where $\tilde{\delta}_{\mathbf{q}} = \sqrt{(1 - 2\lambda_0)}\sqrt{\delta_{\mathbf{q}}^2 - D_{\mathbf{q}}^2}$. The retarded response function $\mathcal{D}_{\mathbf{q}}^{\text{R}}[\omega]$ can be readily found by performing the analytic continuation in e.g. Eq. (13) $i\omega_n \rightarrow \omega + i0^+$. We can now estimate the modification of the critical temperature due to γ using [11]. Again representing the critical temperature as [2] $T_c = \frac{\omega^{\text{ln}}}{1.2} \exp\left\{-1.04 \frac{\lambda_{\text{eff}} + 1}{\lambda_{\text{eff}}}\right\}$. As in the main text the λ_{eff} is essentially related to the (Fermi-surface-averaged) static propagator (12):

$$\begin{aligned} \lambda_{\text{eff}} &= -\nu_0 g_{\text{eff}}^2 \mathcal{D}[0] \\ &= \frac{\lambda_0 (\delta^2 - D^2)}{\gamma^2 + (\delta^2 - D^2)(1 - 2\lambda_0)} \\ &\approx \frac{\lambda_0}{(1 - 2\lambda_0)} \left(1 - \frac{\gamma^2}{(\delta^2 - D^2)(1 - 2\lambda_0)} \right) \end{aligned}$$

Here we omitted the momentum indices as in the main text. Using Eq. 14 we can now estimate the average phonon frequency ω^{ln} as [11]:

$$\begin{aligned} \omega^{\text{ln}} &\equiv \exp \left\{ \frac{\int_0^\infty d\omega \frac{\text{ln} \omega}{\omega} (\rho(x) - \rho(-x))}{\int_0^\infty d\omega \frac{1}{\omega} (\rho(x) - \rho(-x))} \right\} \\ &\approx \tilde{\delta} \exp \left\{ -\frac{\pi\gamma}{2\omega_0} \right\}, \end{aligned}$$

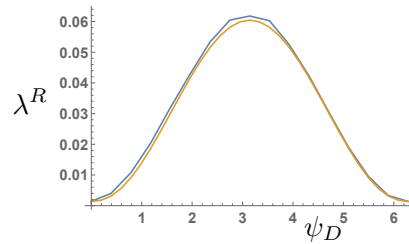


Figure 1. Retarded propagator Eq. (15) as function of the parametric driving phase ψ_D assuming $\psi_\alpha = 0$, $\omega_p = 0.7\omega_0$, $D = 0.15\omega_0$, $\alpha = 0.01\omega_0$: numerical calculation based shown in blue, analytical estimate using high-frequency expansion to lowest order gives $\lambda_0 \approx 4 \frac{g^2 \nu_0}{N} |\bar{a}_0|^2 \mathcal{D}_{\mathbf{q}}(0)$ where $\mathcal{D}_{\mathbf{q}}(0)$ is Eq. (10) and \bar{a}_0 is given in Eq. (8).

where in the last line we expanded in small γ . We thus see that both λ_{eff} and ω^{ln} are decreased in the presence of dissipation which is taken in the form of Lorentz noise.

B. Regime of large detuning

In this section we provide a simple method of determining the pairing strength beyond the high-frequency expansion of the phonon propagator. We start with the model Eq. (5) where the electrons couple to the time-dependent field $\langle \phi_0(t) \rangle \hat{\phi}_{\mathbf{q}}(t)$. Alternatively to the calculation in the main text, the pairing strength can be estimated [3, 8] by taking the static retarded propagator of the corresponding field:

$$\begin{aligned} \lambda^{\text{R}} &\equiv 4 \frac{g^2 \nu_0}{N} \int_{t_0}^{t_0 + 2\pi/\omega_p} dt_0 \int_{t_0}^\infty dt e^{-\epsilon(t-t_0)} \\ &\times \langle \hat{\phi}_0(t) \rangle \langle \hat{\phi}_0(t_0) \rangle \left\langle \left[\hat{\phi}_{\mathbf{q}}(t), \hat{\phi}_{-\mathbf{q}}(t_0) \right] \right\rangle. \quad (15) \end{aligned}$$

Here we averaged over the period of the oscillation of the probe field. λ^{R} is completely equivalent to the bare phonon coupling strength λ_0 computed in the main text. As shown in Fig. 1 the numerical estimate based on Eq. 15 is in good agreement with the high-frequency expansion even when the detuning is not small.

IV. ELIASHBERG EQUATIONS

By denoting $\Sigma[k, i\omega_n] \equiv i\omega_n [1 - Z[k, i\omega_n]]$ in the particle-hole symmetric case and averaging Eqs. (6-8) of the main text over the Fermi surface $\Gamma_n \rightarrow \langle \Gamma_{\mathbf{k}}(\omega_n) \rangle_{\text{FS}}$, $Z_n \rightarrow \langle Z_{\mathbf{k}}(\omega_n) \rangle_{\text{FS}}$ we get [4, 10]:

$$\Gamma_n = \frac{\pi}{\beta} \sum_m \lambda_{\text{eff}} [i\omega_n - i\omega_m] \frac{\Gamma_m}{Z_m |\omega_m|},$$

$$Z_n = 1 + \frac{\pi}{\beta} \sum_m \lambda_{\text{eff}} [i\omega_n - i\omega_m] \left(\frac{\omega_m}{|\omega_m|} \right),$$

where $\lambda_{\text{eff}} [i\omega_n - i\omega_m] = \nu_0 g_{\text{eff}}^2 \mathcal{D} [\vartheta, i\omega_n - i\omega_m]$.

V. MULTIMODE CAVITY OPTOMECHANICS

In this section we show that the effective parametric driving of the out-of-plane phonons Eq. (4) in a multimode cavity optomechanical setting. Mathematical formalism is essentially an extension of [1, 7, 12] to a multimode cavity case. By denoting the annihilation operator of photon field with the transverse momentum \mathbf{k} as $\mathcal{E}_{\mathbf{k}}$, the photon-phonon Hamiltonian reads:

$$H_{\text{int}} = \frac{g_{\text{ph}}}{\sqrt{N}} \sum_{\mathbf{k}, \mathbf{q}} \mathcal{E}_{\mathbf{k}+\mathbf{q}}^\dagger \mathcal{E}_{\mathbf{k}} \phi_{\mathbf{q}} \quad (16)$$

$$H_{\text{phon}} = \sum_{\mathbf{q}} \omega_{\mathbf{q}} a_{\mathbf{q}}^\dagger a_{\mathbf{q}} \quad (17)$$

$$H_{\text{phot}} = \omega_{\text{cav}} \sum_{\mathbf{k}} \mathcal{E}_{\mathbf{k}}^\dagger \mathcal{E}_{\mathbf{k}} + \zeta \left(\mathcal{E}_0 e^{i\Omega_1 t} + \mathcal{E}_0^\dagger e^{-i\Omega_1 t} \right) \quad (18)$$

where g_{ph} is the photon-phonon coupling constant, ζ stands for the cavity driving rate, ω_{cav} is the cavity frequency. In addition, we assume that photons are parametrically driven at the frequency $2\Omega_2$ with the driving strength $\xi_{\mathbf{k}}$:

$$H_{\text{par}} = \sum_{\mathbf{k}} \frac{\xi_{\mathbf{k}}}{2} \left(\mathcal{E}_{\mathbf{k}} \mathcal{E}_{-\mathbf{k}} e^{2i\Omega_2 t} + \mathcal{E}_{\mathbf{k}}^\dagger \mathcal{E}_{-\mathbf{k}}^\dagger e^{-2i\Omega_2 t} \right) \quad (19)$$

We now linearize the interaction Hamiltonian assuming the driving is strong enough and get:

$$H_{\text{int}} = \frac{\zeta_{\text{ss}} g_{\text{ph}}}{\sqrt{N}} \sum_{\mathbf{q}} \left(\mathcal{E}_{\mathbf{q}}^\dagger e^{-i\Omega_1 t} + \mathcal{E}_{-\mathbf{q}} e^{i\Omega_1 t} \right) \phi_{\mathbf{q}} \quad (20)$$

where the mean-field cavity coherence is $\langle \mathcal{E}_0 \rangle \approx \zeta_{\text{ss}} e^{-i\Omega_1 t}$ with $\zeta_{\text{ss}} = -\zeta / (\omega_{\text{cav}} - \Omega_1)$. We now transform to the interaction picture: $\mathcal{E}_{\mathbf{q}} \rightarrow \mathcal{E}_{\mathbf{q}} e^{-i\Omega_2 t}$ and $a_{\mathbf{q}} \rightarrow a_{\mathbf{q}} e^{i(\Omega_2 - \Omega_1)t}$ and neglect all rotating terms:

$$H_{\text{int}} \approx \frac{\zeta_{\text{ss}} g_{\text{ph}}}{\sqrt{N}} \sum_{\mathbf{q}} \left(\mathcal{E}_{\mathbf{q}}^\dagger a_{\mathbf{q}} + \mathcal{E}_{\mathbf{q}} a_{\mathbf{q}}^\dagger \right) \quad (21)$$

$$H_{\text{phon}} = \sum_{\mathbf{q}} \delta_{\mathbf{q}} a_{\mathbf{q}}^\dagger a_{\mathbf{q}} \quad (22)$$

$$H_{\text{phot}} = \Delta_{\text{cav}} \sum_{\mathbf{k}} \mathcal{E}_{\mathbf{k}}^\dagger \mathcal{E}_{\mathbf{k}} \quad (23)$$

$$H_{\text{par}} = \sum_{\mathbf{k}} \frac{\xi_{\mathbf{k}}}{2} \left(\mathcal{E}_{\mathbf{k}} \mathcal{E}_{-\mathbf{k}} + \mathcal{E}_{\mathbf{k}}^\dagger \mathcal{E}_{-\mathbf{k}}^\dagger \right) \quad (24)$$

where $\Delta_{\text{cav}} = \omega_{\text{cav}} - \Omega_2$ and $\delta_{\mathbf{q}} = (\omega_{\mathbf{q}} - \Omega_1 + \Omega_2)$. Adiabatic elimination of cavity modes. We First do the Bogolyubov transform: $\mathcal{E}_{\mathbf{k}} = u_{\mathbf{q}} \gamma_{\mathbf{q}}^\dagger + v_{\mathbf{q}} \gamma_{-\mathbf{q}}$ and $\Lambda_{\mathbf{k}} = \sqrt{\Delta_{\text{cav}}^2 - \xi_{\mathbf{k}}^2}$. And second, we eliminate $\gamma_{\mathbf{q}}$ assuming it is in vacuum state. For $\Lambda_{\mathbf{k}} \gg \delta_{\mathbf{q}}$:

$$H_{\text{phon}} = \sum_{\mathbf{q}} \left(\delta_{\mathbf{q}} + \frac{\Delta_{\text{cav}}}{\Delta_{\text{cav}}^2 - \xi_{\mathbf{q}}^2} \frac{\zeta_{\text{ss}}^2 g_{\text{ph}}^2}{N} \right) a_{\mathbf{q}}^\dagger a_{\mathbf{q}} + \frac{\zeta_{\text{ss}}^2 g_{\text{ph}}^2}{2N} \sum_{\mathbf{q}} \frac{\xi_{\mathbf{q}}}{\Delta_{\text{cav}}^2 - \xi_{\mathbf{q}}^2} \left(a_{-\mathbf{q}} a_{\mathbf{q}} + a_{\mathbf{q}}^\dagger a_{-\mathbf{q}}^\dagger \right)$$

A. Squeezing by light modulation

In this section we consider the multimode membrane squeezing by light modulation in analogy to [9]. We start with the Hamiltonian Eqs. (16-18):

$$H_{\text{int}} = \frac{g_{\text{ph}}}{\sqrt{N}} \sum_{\mathbf{k}, \mathbf{q}} \mathcal{E}_{\mathbf{k}+\mathbf{q}}^\dagger \mathcal{E}_{\mathbf{k}} \phi_{\mathbf{q}} \quad (25)$$

$$H_{\text{phon}} = \sum_{\mathbf{q}} \omega_{\mathbf{q}} a_{\mathbf{q}}^\dagger a_{\mathbf{q}} \quad (26)$$

$$H_{\text{phot}} = \omega_{\text{cav}} \sum_{\mathbf{k}} \mathcal{E}_{\mathbf{k}}^\dagger \mathcal{E}_{\mathbf{k}} + \left(\zeta^*(t) \mathcal{E}_0 e^{i\Omega_1 t} + \zeta(t) \mathcal{E}_0^\dagger e^{-i\Omega_1 t} \right) \quad (27)$$

We note that the driving Hamiltonian is now time dependent $\zeta(t)$. We now transform into the rotating frame with respect to the laser carrier frequency Ω_1 and displace the cavity variables by their mean-field value: $\mathcal{E}_0 \rightarrow \frac{\zeta(t)}{\Delta_{\text{cav}}} + \mathcal{E}_0$, where $\Delta_{\text{cav}} = \Omega_1 - \omega_{\text{cav}}$. Assuming Δ_{cav} is the largest energy scale we find:

$$H_{\text{int}} \approx \frac{g_{\text{ph}}}{\Delta_{\text{cav}} \sqrt{N}} \sum_{\mathbf{q}} \left(\mathcal{E}_{\mathbf{q}}^{\dagger} \zeta(t) + \zeta^*(t) \mathcal{E}_{-\mathbf{q}} \right) \left(a_{\mathbf{q}} + a_{-\mathbf{q}}^{\dagger} \right) \quad (28)$$

$$+ \frac{g_{\text{ph}}}{\sqrt{N}} \frac{|\zeta(t)|^2}{\Delta_{\text{cav}}^2} \phi_0 \quad (29)$$

$$(30)$$

$$H_{\text{phon}} = \sum_{\mathbf{q}} \omega_{\mathbf{q}} a_{\mathbf{q}}^{\dagger} a_{\mathbf{q}} \quad (31)$$

$$H_{\text{phot}} = -\Delta_{\text{cav}} \sum_{\mathbf{k}} \mathcal{E}_{\mathbf{k}}^{\dagger} \mathcal{E}_{\mathbf{k}} \quad (32)$$

We now assume $\zeta(t) = \zeta \cos((\omega_0 - \xi)t)$, where $\omega_0 = \omega_{\mathbf{q}=0}$ and ξ is some frequency offset that we identify below. We transform phonon modes into rotating frame with respect to $\omega_0 - \xi$:

$$H_{\text{int}} \approx \frac{g_{\text{ph}} \zeta}{2\Delta_{\text{cav}} \sqrt{N}} \sum_{\mathbf{q}} \left(\mathcal{E}_{\mathbf{q}}^{\dagger} \left(a_{\mathbf{q}} + a_{-\mathbf{q}}^{\dagger} \right) + \mathcal{E}_{-\mathbf{q}} \left(a_{\mathbf{q}} + a_{-\mathbf{q}}^{\dagger} \right) \right) \quad (33)$$

$$(34)$$

$$(35)$$

$$H_{\text{phon}} = \sum_{\mathbf{q}} (\omega_{\mathbf{q}} - \omega_0 + \xi) a_{\mathbf{q}}^{\dagger} a_{\mathbf{q}} \quad (36)$$

$$H_{\text{phot}} = -\Delta_{\text{cav}} \sum_{\mathbf{k}} \mathcal{E}_{\mathbf{k}}^{\dagger} \mathcal{E}_{\mathbf{k}} \quad (37)$$

We now eliminate the cavity mode and find:

$$H_{\text{int}} \approx -\frac{g_{\text{ph}}^2 \zeta^2}{4\Delta_{\text{cav}}^3} \sum_{\mathbf{q}} \left(a_{\mathbf{q}} + a_{-\mathbf{q}}^{\dagger} \right) \left(a_{-\mathbf{q}} + a_{\mathbf{q}}^{\dagger} \right) \quad (38)$$

$$H_{\text{phon}} = \sum_{\mathbf{q}} (\omega_{\mathbf{q}} - \omega_0 + \xi) a_{\mathbf{q}}^{\dagger} a_{\mathbf{q}} \quad (39)$$

We now see that upon choosing $\xi = \frac{g_{\text{ph}}^2 \zeta^2}{4\Delta_{\text{cav}}^3}$ we can parametrically excite the phonon mode.

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