# Maximal violation of Bell inequalities using continuous-variable measurements

Jérôme Wenger, Mohammad Hafezi, Frédéric Grosshans, Rosa Tualle-Brouri, and Philippe Grangier*
*Laboratoire Charles Fabry de l'Institut d'Optique, UMR 8501 du CNRS, F91403 Orsay, France*
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We propose a whole family of physical states that yield a maximal violation of Bell inequalities, when using quadrature-phase homodyne detection. This result is based on a binning process called root binning, that is used to transform the continuous-variable measurements into binary results needed for the tests of quantum mechanics versus local realistic theories. A physical process in order to produce such states is also suggested. The use of high-efficiency spacelike separated homodyne detections with these states and this binning process would result in a conclusive loophole-free test of quantum mechanics.

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## I. INTRODUCTION

Nonseparability, or entanglement, has emerged as one of the most striking feature of quantum mechanics. In 1935, it led Einstein, Podolsky, and Rosen to suggest [1] that quantum mechanics is incomplete, on the premise that any physical theory of nature must be both "local" and "realistic." To quantify the debate between quantum mechanics and local realistic (classical) theories, Bell introduced a set of inequalities that must be obeyed by any local realistic theory whereas they are violated by quantum mechanics [2–4]. These results shifted the debate from the realm of philosophy to experimental physics. The experiments done at the beginning of the 1980s by Aspect and co-workers [5–7] convincingly supported the predictions of quantum mechanics, but admittedly left open two so-called "loopholes," that have to be addressed for the evidence to be fully conclusive.

The first of these loopholes, called "locality" loophole, arises when the separation between the measured states is not large enough to completely discard the exchange of subluminal signals during the measurements. The second loophole, called detection-efficiency loophole, occurs when the particle detectors are inefficient enough so that the detected events may be unrepresentative of the whole ensemble. In 1998, Weihs *et al.* [8] achieved communication-free condition by using a type-II parametric down-conversion source and fast random switching of the analyzers, that were separated by about 400 m. This closed the locality loophole, but their detection efficiency was not sufficient to close the second loophole. In 2001, Rowe *et al.* [9] measured quantum correlations between two entangled beryllium ions with up to 80% overall efficiency, closing the detection-efficiency loophole, but unfortunately the ions were too close (about 3 $\mu$m) to avoid the locality loophole. Hence a present challenge is to design and perform an experiment that closes both loopholes to lead to a full logically consistent test of any local realistic theory.

Quantum optics suggests good candidates, as photons can be transported to sufficient long distances to avoid the locality loophole. To close the detection-efficiency loophole, an alternative to photon-counting schemes consists in quadrature-phase homodyne measurements, that use strong local oscillators detected by highly efficient photodiodes. Up to date, a few theoretical proposals that use quadrature-phase homodyne detections have been made [10–13] but for these setups the Bell inequality violation is a few percents only, that lies far away from the maximal violation attainable : $2\sqrt{2}$ (compared to a classical maximum of 2) for the Clauser-Horne-Shimony-Holt (CHSH) inequality [3], and $(1+\sqrt{2})/2$ (compared to a classical maximum of 1) for the Clauser-Horne (CH) inequality [4]. Gilchrist *et al.* [10,11] use a *circle* or pair coherent state produced by nondegenerate parametric oscillation with the pump adiabatically removed. This state leads to a theoretical violation of about 1.015 $(>1)$ of the CH Bell inequality. Munro [13] considers correlated photon number states of the form

$$|\Psi\rangle=\sum_{n=0}^{N} c_n|n\rangle|n\rangle, \tag{1}$$

where $N$ is truncated at $N=10$. He then performs a numerical optimization on each $c_n$ coefficient to maximize the violation of the CHSH Bell inequality when an homodyne measurement is performed. For this specific state, the CHSH inequality is violated by 2.076 $(>2)$ and the CH inequality by 1.019 $(>1)$. From a different phase-space approach, Auberson *et al.* [14] derive phase space Bell inequalities and propose a state that yields a maximal violation of up to $2\sqrt{2}$ $(>2)$. This state can be expressed in the position space by

$$\Psi_{\pm}(q_1,q_2)=\frac{1}{2\sqrt{2}}[1\pm e^{i(\pi/4)}\mathrm{sgn}(q_1)\mathrm{sgn}(q_2)]f(|q_1|)f(|q_2|), \tag{2}$$

where $f(q)$ is a regularized form of $(1/\sqrt{q})$, with $\int_{-\infty}^{+\infty}dq\, f(q)^2=1$. The main problem with this wave function lies in its singularities and phase switches. Therefore, it requires nontrivial regularization procedures to be considered as a suitable physical state. Following these various attempts, we are thus looking for a "simple" physical state that would lead to a maximal violation of a Bell inequality.

In this paper, we consider the CHSH Bell inequality [3] (sometimes referred to as the *spin* inequality). Figure 1 depicts an idealized setup for a general Bell inequality measurement. Two entangled substates are viewed by two ana-

*Email address: philippe.grangier@iota.u-psud.fr

 

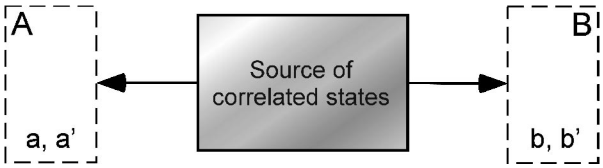


FIG. 1. Schematic of a generalized Bell experiment. The source generates correlated states that are directed to the $A$ and $B$ devices used to perform the measurements, with adjustable parameters $a$ and $b$. Each measurement provides a binary result "+" or "−" individually.

lyzers and detectors at locations $A$ and $B$, where $a$ and $b$ denote any adjustable parameter at $A$ and $B$. In our particular case, we will use quadrature-phase homodyne measurements, which could have an efficiency high enough to close the detection-efficiency loophole. Moreover, the apparatuses $A$ and $B$ can be in principle spacelike separated, thereby excluding action at distance, and closing also the locality loophole. We point out that in the present approach all the detected light has to be taken into account, i.e., the relevant signal is the photocurrent generated by the interferometric mixing and photodetection of the local oscillator and input quantum state. Therefore, no "supplementary assumption" [3] will be needed to interpret the data. Under these conditions, the CHSH Bell inequality can be written [3] as

$$S=|E(a',b')+E(a',b)+E(a,b')-E(a,b)|\leqslant 2, \quad (3)$$

where the correlation function $E(a,b)$ is given by

$$E(a,b)=P_{++}(a,b)+P_{--}(a,b)-P_{+-}(a,b)-P_{-+}(a,b) \quad (4)$$

with $P_{++}(a,b)$ the probability that a "+" occurs at both $A$ and $B$, given $a$ and $b$.

In this paper, we propose explicitly a set of physical states that yield a violation of the CHSH inequality with a value of $S$ arbitrarily close to $2\sqrt{2}$, when measured by an ideal quadrature-phase homodyne detection. In Sec. II, we describe how we convert the continuous quadrature amplitude into a binary result "+" or "−" for each apparatus $A,B$ using a process called *Root Binning*. In Sec. III, a specific state that yields a large violation of the CHSH inequality is presented. This state is generalized in Sec. IV to derive a whole family of states that violate this Bell inequality. The issue of preparing such states is addressed in Sec. V, and various other theoretical and practical issues are briefly discussed in the conclusion.

## II. ROOT BINNING

To begin our study, we consider a state of the form of a superposition of two two-particles wave functions with a relative phase.

$$|\Psi\rangle=\frac{1}{\sqrt{2}}(|ff\rangle+e^{i\theta}|gg\rangle) \qquad 0\leqslant\theta\leqslant 2\pi, \quad (5)$$

with $f$ real, even, and normalized to unity while $g$ is assumed real, odd, and normalized to unity. This kind of state looks similar to the one that used by Auberson *et al.* [14], but the $f$ and $g$ functions will be quite different as well as the binning of the continuous variables being measured.

The quadrature-phase homodyne measurement outputs a continuous variable, but using the CHSH inequality for testing local realism versus quantum mechanics require a binary result. Hence for a given quadrature measurement $q_i$ ($i=1,2$) at either location $A$ or $B$, we need to classify the result as either "+" or "−." In Refs. [11,13,14], the "positive-negative" binning is used, that is the result is classified as "+" if $q_i\geqslant 0$ and "−" if $q_i<0$.

The choice of binning is quite arbitrary. For state (5) we can consider another type of binning, we call *root binning*, that depends on the roots of the functions $f$ and $g$ (that are known in advance to the experimenters). We assign "+" when the result $q_i$ lies in an interval where $f(q)$ and $g(q)$ have the same sign, and "−" if $q_i$ is in an interval where $f$ and $g$ have opposite signs. We define $D^+$ as the union of the intervals in which $f(q)$ and $g(q)$ have the same sign and $D^-$ as the union of the intervals in which $f(q)$ and $g(q)$ have opposite signs. We have thus

$$D^+=\{\forall q\epsilon\mathbb{R}\,|\,f(q)g(q)\geqslant 0\}, \quad (6)$$

$$D^-=\{\forall q\epsilon\mathbb{R}\,|\,f(q)g(q)<0\}. \quad (7)$$

Let us first consider the case when quadrature measurements in position space have been performed on both sides. So the binary probabilities we need for the CHSH type of Bell inequality will be

$$P_{++}=\int_{D^+}\int_{D^+}dq_1dq_2P(q_1,q_2), \quad (8)$$

$$P_{+-}=\int_{D^+}\int_{D^-}dq_1dq_2P(q_1,q_2), \quad (9)$$

$$P_{-+}=\int_{D^-}\int_{D^+}dq_1dq_2P(q_1,q_2), \quad (10)$$

$$P_{--}=\int_{D^-}\int_{D^-}dq_1dq_2P(q_1,q_2), \quad (11)$$

with

$$P(q_1,q_2)=|\langle q_1|\langle q_2|\Psi\rangle|^2 \quad (12)$$

$$=\frac{1}{2}[f(q_1)^2f(q_2)^2+g(q_1)^2g(q_2)^2 + 2\cos\theta f(q_1)g(q_1)f(q_2)g(q_2)]. \quad (13)$$

The correlation fuction for the state $|\Psi\rangle$ is given by

$$E_{q_1,q_2}=P_{++}+P_{--}-P_{+-}-P_{-+} \quad (14)$$

and as we have chosen $f$ even and $g$ odd, we get the remarkably simple expression,

$$E_{q_1,q_2}=V^2\cos\theta, \tag{15}$$

where

$$V=\int_{D^+}f(q)g(q)dq-\int_{D^-}f(q)g(q)dq$$
$$=\int_{-\infty}^{+\infty}|f(q)g(q)|dq. \tag{16}$$

A similar binning will be applied for the momentum part. Since we suppose that $f(q)$ is a real and even function while $g(q)$ is a real and odd, $f(q)$ has a real even Fourier transform $\tilde{f}(p)$ while $g(q)$ has an imaginary Fourier transform $i\tilde{h}(p)$, where $\tilde{h}(p)$ is a real and odd function. Using these properties, and taking care of the supplementary $i$ factor, the same reasoning applies for $\tilde{f}$ and $\tilde{h}$ as for $f$, $g$. Denoting as $D'^+$ and $D'^-$ the intervals associated with $\tilde{f}$ and $\tilde{h}$, we obtain

$$E_{p_1,p_2}=-W^2\cos\theta, \tag{17}$$

where

$$W=\int_{D'^+}\tilde{f}(p)\tilde{h}(p)dp-\int_{D'^-}\tilde{f}(p)\tilde{h}(p)dp$$
$$=\int_{-\infty}^{+\infty}|\tilde{f}(p)\tilde{h}(p)|dp, \tag{18}$$

and equivalently,

$$E_{q_1,p_2}=-V\ W\ \sin\theta, \tag{19}$$

$$E_{p_1,q_2}=-V\ W\ \sin\theta. \tag{20}$$

Hence by combining Eqs. (15), (17), (19), and (20) we can write the CHSH inequality (3)

$$S=|\cos(\theta)(V^2+W^2)-2\sin(\theta)VW|\leqslant 2. \tag{21}$$

The maximum of $S$ with respect to $\theta$ is obtained for $\tan\theta_m=-2VW/(V^2+W^2)$, and we have $\theta_m\to-\pi/4$ as $V,W\to 1$. For this optimized $\theta_m$ we get the Bell inequality

$$S=|\sqrt{W^4+V^4+6V^2W^2}|\leqslant 2. \tag{22}$$

Using this really simple expression (22), the debate between quantum mechanics and local realistic theories boils down to find functions $f$ and $g$ such that the integrals $V$, $W$ violate (22). An interesting feature appears when the distributions are eigenstates of the Fourier transform, so that $V=W$ and Eq. (22) becomes

$$S=2\sqrt{2}\ \ V^2\leqslant 2. \tag{23}$$

So if such functions have the right overlap needed to obtain $V=1$, one will get the maximal violation of the above inequality, which is obtained for $S=2\sqrt{2}$.

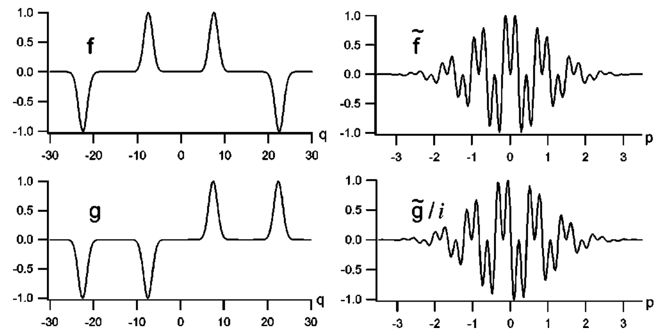


FIG. 2. $f$ and $g$ for a four-peaks state described by Eqs. (26) and (27) in position space (left) and momentum space (right), with $a=7.5$ ($\alpha=15$). Left axes are in arbitrary units and normalized to unity.

When compared to the positive-negative binning, root binning has the advantage of having two parameters $V$ and $W$ to play with while the positive-negative binning has only one [14]. Moreover, as we will show now, the above Bell inequality is violated by simple wave functions, that no longer have the singularities that appeared in Ref. [14].

## III. LINEAR SUPERPOSITION OF FOUR COHERENT STATES

In order to propose an explicit expression of a state that violates the Bell inequality (22), let us first consider the case of a superposition of two coherent states of amplitudes $a$ and $-a$. This state, sometimes referred to as a Schrödinger cat, involves intrinsic quantum features such as negative Wigner functions, which make it an interesting candidate for our state $|f\rangle$ or $|g\rangle$. We must choose an even wave function for $f(q)$ and an odd for $g(q)$,

$$f(q)\propto e^{-(q+a)^2/2}+e^{-(q-a)^2/2}, \tag{24}$$

$$g(q)\propto -e^{-(q+a)^2/2}+e^{-(q-a)^2/2}, \tag{25}$$

unfortunately for this simple state we get $V=1$ and $W\simeq 0.64$ for $a\to\infty$, so that $S\simeq 1.90<2$. Therefore this state cannot be used for violating Bell inequality (note that Gilchrist *et al.* [10,11] also consider similar states, but without getting a violation of Bell inequalities).

Instead of superpositions of two coherent states, we consider quantum superpositions that have four Gaussian components. Let us for instance consider

$$f(q)\propto -e^{-(q+3a)^2/2}+e^{-(q+a)^2/2}+e^{-(q-a)^2/2}-e^{-(q-3a)^2/2}, \tag{26}$$

$$g(q)\propto -e^{-(q+3a)^2/2}-e^{-(q+a)^2/2}+e^{-(q-a)^2/2}+e^{-(q-3a)^2/2}. \tag{27}$$

The functions $f(q)$ and $g(q)$ are depicted in Fig. 2 together with their Fourier transform. We note that the distance between each peak and its neighbors is $\alpha=2a$. This disposition yields an optimal overlap of $\tilde{f}$ and $\tilde{g}$ and thus a high value of $S$. The best violation appears when the peaks move off as

TABLE I. $S$ for $N$-peaks states defined by Eqs. (28) and (29) and $\alpha=15$, each peak having the same height.

| $N$ | 2 | 4 | 6 | 8 | 10 | 12 |
|---|---|---|---|---|---|---|
| $S$ | 1.895 | 2.417 | 2.529 | 2.611 | 2.649 | 2.681 |

$a\to\infty$. In that case, $V=1$, $W=8/(3\pi)$ and thus we get the significant violation of $S-2\simeq 0.417$ (in facts, the condition $a\to\infty$ appeared to be not so strict numerically, as an amplitude $a=5$ is enough to obtain $S\simeq 2.417$). Such a violation represents a large improvement compared to Munro's best violation of 0.076 [13], for a state with no singularity and at least as easy to produce as Munro's $c_n$ optimized state (1). However, we are still away from the maximal value $2\sqrt{2}$ of the CHSH Bell inequality [15]. In the following section, we will propose a set of states to get closer to the maximal violation.

## IV. LINEAR SUPERPOSITION OF *N* COHERENT STATES

The result obtained with the four Gaussian component states suggests that a way to get a stronger violation is to increase the total number $N$ of peaks of the states $|f_N\rangle$ and $|g_N\rangle$, with the proper sign between the peaks. We thus define, for a given amplitude $\alpha$

$$f_{N;\alpha}(q)\propto\sum_{j=-N/2}^{(N/2)-1}\cos\left(\frac{\pi}{4}[2j+1]\right)\exp\{-(q-[j+\tfrac{1}{2}]\alpha)^2/2\},\qquad(28)$$

$$g_{N;\alpha}(q)\propto\sum_{j=-(N/2)}^{(N/2)-1}\sin\left(\frac{\pi}{4}[2j+1]\right)\exp\{-(q-[j+\tfrac{1}{2}]\alpha)^2/2\}.\qquad(29)$$

Table I presents the results of the calculation of $S$ according to formula (22) for the state defined by Eqs. (5), (28), and (29). As expected, the quantity $S$ increases with the number of peaks and tends to $2\sqrt{2}$. To prove this point, let us consider the two following distributions that have an infinite number of peaks. These distributions are depicted with their Fourier transform in Fig. 3.

$$f_{\infty;\alpha}(q)\propto\sum_{j=-\infty}^{+\infty}\delta(q-\alpha(j+1/2))\cos\left(\frac{\pi q}{2\alpha}\right),\qquad(30)$$

$$g_{\infty;\alpha}(q)\propto\sum_{j=-\infty}^{+\infty}\delta(q-\alpha(j+1/2))\sin\left(\frac{\pi q}{2\alpha}\right).\qquad(31)$$

Up to scaling factors, these distributions appear to be almost identical to their Fourier transform, while for the specific amplitude $\alpha=\sqrt{\pi}$ they are exact eigenstates of the Fourier transform. Moreover, as $f_\infty$ and $g_\infty$ overlap perfectly, these distributions would yield the maximal violation of the CHSH inequality, as $S=2\sqrt{2}$ from Eq. (23). Of course, they are unphysical (non-normalizable) states, but the wave functions $f_N$ and $g_N$ in Eqs. (28) and (29) can be considered as regularized forms of $f_\infty$ and $g_\infty$, widening the Dirac delta

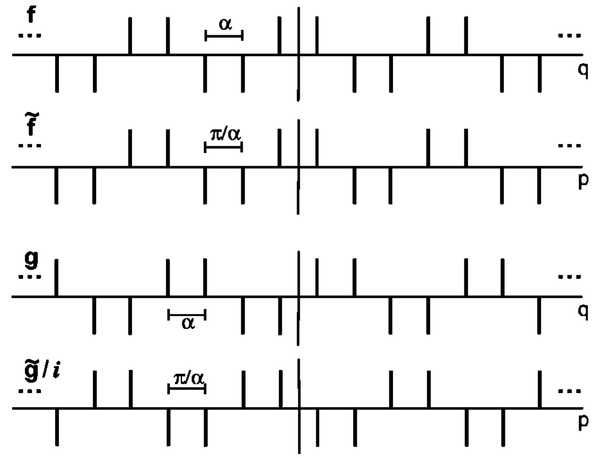


FIG. 3. Infinite-peaks state, represented in the position and momentum phase space. Thick segments denote Dirac delta functions.

functions to Gaussians and taking a finite number of elements. Thus one can understand that $S\to 2\sqrt{2}$ as $N\to\infty$.

Another regularization of wave functions (30) and (31) consists in a widening of the Dirac functions to Gaussians of width $s$ associated with a Gaussian envelope of width $1/s$:

$$f_{\infty;\alpha,s}(q)\propto G_{1/s}(q)[f_{\infty;\alpha}*G_s(q)],\qquad(32)$$

$$g_{\infty;\alpha,s}(q)\propto G_{1/s}(q)[g_{\infty;\alpha}*G_s(q)],\qquad(33)$$

where $G_s(q)=\exp(-q^2/2s^2)$, $*$ denotes the convolution product, and $s$ is a squeezing parameter. When $s\ll 1$, one indeed has

$$\tilde{f}_{\infty;\alpha,s}(p)\propto G_s*[f_{\infty;\pi/\alpha}G_{1/s}](p)\approx f_{\infty;\pi/\alpha,s}(p),\qquad(34)$$

$$\tilde{g}_{\infty;\alpha,s}(p)\propto G_s*[g_{\infty;\pi/\alpha}G_{1/s}](p)\approx g_{\infty;\pi/\alpha,s}(p).\qquad(35)$$

The value of $S$ exhibits a symmetry $S(\alpha)=S(\pi/\alpha)$, with a maximum reached for $\alpha=\sqrt{\pi}$, where $f$ and $g$ are approximately eigenstates of the Fourier transform.

Thanks to the Gaussian envelope, the above functions can be truncated to a finite total number of peaks $N$ without affecting numerically the $f$ and $g$ functions, provided

$$N>\frac{2\sqrt{2|\ln\varepsilon|}}{\alpha s},\qquad(36)$$

where $\varepsilon$ is an arbitrary small tolerance parameter. For $\varepsilon=0.01$, $\alpha=\sqrt{\pi}$, and $s=0.3$, the condition yields $N\geqslant 12$. Given these parameters and $N=12$, we get $S\approx 2\sqrt{2}$ with a relative error of 0.01%. Regardless to this condition, one may also arbitrarily choose to limit the above functions to $N$ peaks for given parameters $s$ and $\alpha$, but as such $f$ and $g$ cannot directly be considered as truncated $f_{\infty;\alpha,s}$ and $g_{\infty;\alpha,s}$, the best amplitude will differ from $\sqrt{\pi}$ and $S$ will slightly move away from $2\sqrt{2}$. Some results obtained for $s=0.3$ are presented on Table II for $N$ running from 4 to 12. The value of $\alpha$ used in this table is numerically calculated in order to maximize $S$. Here the violation is considerably improved

TABLE II. $S$ as a function of the number of peaks $N$ for a squeezing parameter $s=0.3$ and a Gaussian envelope of width $1/s$, for an optimized amplitude $\alpha_{opt}$.

| $N$ | 4 | 6 | 8 | 10 | 12 |
|---|---|---|---|---|---|
| $\alpha_{opt}$ | 2.6 | 2.3 | 2 | 1.8 | 1.8 |
| $S$ | 2.764 | 2.823 | 2.826 | 2.828 | 2.828 |

compared to Table I, with for instance $S=2.764$ with $N=4$, and $S=2.828$ with $N=10$. In Fig. 4 we display $f(q)$ and $g(q)$ for the case $N=12$ showing that these functions are nearby self Fourier transform.

In Appendix A, we rewrite these states on the Fock-states basis and express the $f$ and $g$ functions as combinations of Hermite polynomials.

## V. CONDITIONAL PREPARATION OF ENTANGLED SUPERPOSITIONS OF $N$ COHERENT STATES

Preparing an entangled superposition of $N$ coherent or squeezed states is a challenging task, we begin by focusing our attention on how one single $N$ Gaussian components state as defined by Eq. (29) could be generated. This state has strong similarities with the *encoded states* introduced by Gottesman and co-workers [16] to perform quantum error-correction codes. Recently, Travaglione and Milburn [17] presented a proposal to generate nondeterministically such *encoded states*. Following this study, we will first show how to generate the state $|g\rangle$ by applying a specific sequence of operations similar to Ref. [17] and then derive a setup to produce the whole state (5).

The preparation procedure begins with the quantum system in the vacuum state $|0\rangle$ and an ancilla qubit in the ground state $|0\rangle_a$ (if some squeezing parameter is needed, one may take a squeezed vacuum as quantum system and follow the procedure we describe here. For simplicity reasons, we set $s=1$). Let us first apply

$$He^{-i\alpha p\sigma_z}H, \tag{37}$$

with $p$ is the momentum operator applied to the continuous variable state, $H$ is the Hadamard gate, and $\sigma_z$ is the Pauli matrix applied to the qubit:

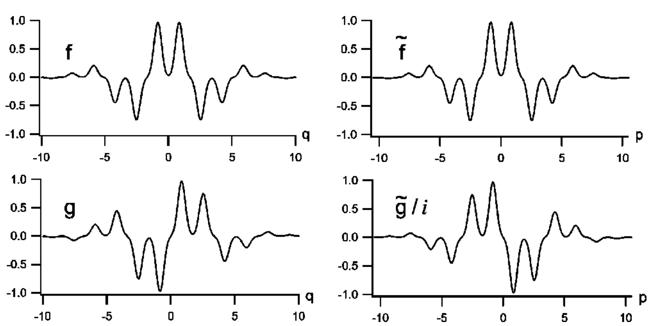


FIG. 4. $N=12$ states described by Eqs. (32) and (33) presented in position space (left) and in momentum space (right) for parameters $\alpha=1.8$ and $s=0.3$. Left axes are in arbitrary units and normalized to unity.

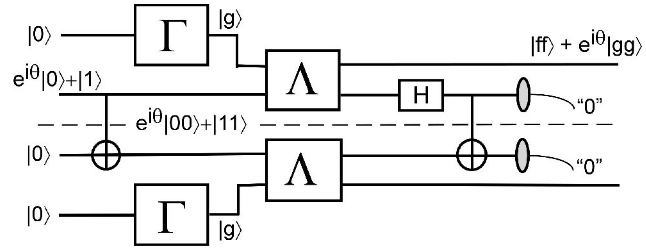


FIG. 5. Schematic of the setup used to generate the state $|\Psi\rangle=(1/\sqrt{2})(|ff\rangle+e^{i\theta}|gg\rangle)$ with $f$ and $g$ defined by Eqs. (28) and (29). See text for the notations.

$$H=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}, \qquad \sigma_z=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{38}$$

We then have a probability 1/2 of measuring the qubit either in the excited or in the ground state. If it is found in the $|1\rangle_a$ state, the continuous variable is left in the state $|\Upsilon_1\rangle \propto -|-\alpha\rangle+|\alpha\rangle$, otherwise the procedure is stopped and we try again. The qubit is then bit flipped to $|0\rangle_a$ and we go on by applying the sequence

$$He^{-i2\alpha p\sigma_z}H. \tag{39}$$

Measuring the qubit in the $|0\rangle_a$ results in the continuous variable left in the state $|\Upsilon_2\rangle \propto -|-3\alpha\rangle+|-\alpha\rangle-|\alpha\rangle+|3\alpha\rangle$. To increase the number of peaks, we iterate the following procedure given $|\Upsilon_{n-1}\rangle$ and the qubit in $|0\rangle_a$:

(1) Apply the operators

$$He^{-i2^{n-1}\alpha p\sigma_z}H. \tag{40}$$

(2) Measure the qubit.

(3) If the qubit was in the state $|0\rangle_a$, we have created $|\Upsilon_n\rangle$.

(4) If the qubit was in the state $|1\rangle_a$, discard and try again.

Once the number of peaks is considered satisfactory, we stop the previous iteration. The last point to generate $|g\rangle$ is to apply the following sequence:

$$He^{-i(\alpha/2)p\sigma_z}H. \tag{41}$$

If the qubit is found in the state $|0\rangle_a$, we have created the state $|g\rangle$ as defined in Eq. (29). If the protocol was stopped after $n$ iterations (corresponding to $|\Upsilon_n\rangle$), the state $|g\rangle$ is created with probability $1/2^{n+1}$ and shows $N=2^{n+1}$ peaks. In the last section of Ref. [17], Travaglione and Milburn briefly consider the question of the physical implementation of this iteration process using a radio-frequency ion trap. We refer the reader to this paper for further details.

For the creation process of the state $|\Psi\rangle=(1/\sqrt{2})(|ff\rangle+e^{i\theta}|gg\rangle)$, we present in Fig. 5 a global view of our scheme. The $\Gamma$-labeled box corresponds to the generation of $|g\rangle$ we just saw. Our process is based on a controlled-NOT gate that entangles two qubits and thus produces the state $|\psi\rangle_a=1/\sqrt{2}(|11\rangle_a+e^{i\theta}|00\rangle_a)$. Each qubit is then associated with a state $|g\rangle$ through the following operator

$$\Lambda = \exp\left[\frac{i\pi}{4}\left(\frac{q}{\alpha/2} - 1\right)(1-\sigma_z)\right], \tag{42}$$

where $q$ is the position operator of the $|g\rangle$ state and $\sigma_z$ is the Pauli matrix applied to the qubit. If the qubit is at zero, the $|g\rangle$ state will be left unchanged. If the qubit is at one, for $s \ll \alpha$ the sign of each two peak is changed so that the $|g\rangle$ state is changed into the $|f\rangle$ state. After this operator $\Lambda$ has been applied, the whole state is left in

$$e^{i\theta}|gg00\rangle + |ff11\rangle. \tag{43}$$

In order to disentangle the two qubits from the continuous part, one has to make the qubits pass through an Hadamard plus a controlled-NOT gate. Finally measuring one qubit in the zero state will project the continuous variable system onto the awaited state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|ff\rangle + e^{i\theta}|gg\rangle). \tag{44}$$

Such a process generates the linear superpositon of $N$ coherent states as defined in Eqs. (28) and (29), where each component has the same height. The case (32) and (33) with a Gaussian envelope is more complicated to produce because each peak must be separately weighted, but as seen in Table I, the violation for a state (28), (29) with equal heights is already quite strong ($S \simeq 2.41$ for $N=4$).

## VI. CONCLUSION

Considering the quadrature-phase homodyne detection, we have derived a different binning process called *root binning* to transform the continuous variables measured into binary results to be used in the test of quantum mechanics versus local realistic theories. For this process, we propose a whole family of physical states that yield a violation arbitrarily close to the maximal violation in quantum mechanics and much stronger than previous works in the domain [10–13].

We have also tested root binning on other interesting forms of Bell inequality that are the information-theoretic inequalities developed by Braunstein and Caves [18] and generalized by Cerf and Adami [19]. Using quadrature measurements, root binning and our state defined by Eqs. (5), (32), and (33), we unfortunately could find no violation for neither Braunstein's nor Cerf's form of information-theoretic Bell inequality. Our state in fact tends to the minimum limit for violation of these information inequalities when $V \to 1$ and $W \to 1$. As a matter of fact, the binning process discards a lot of information that lies within each interval of the binning. This information loss may prevent any violation of information-theoretic inequalities.

As a conclusion, let us point out that though the present idea sounds quite attractive, its practical implementation is very far fetched. Though we do propose a theoretical scheme to prepare the required states, it relies on various features (coupling Hamiltonian, controlled-NOT gates) that are not presently available with the required degree of efficiency. For going in more details into the implementation of the present scheme, inefficiencies and associated decoherence effects should be examined in detail for each required step, i.e., preparation, propagation, and detection. Also, various possible implementations of the proposed scheme should be considered [9,17,20,21]. Such a study is out of the scope of the present paper, that has mostly the goal to show that arbitrarily high violations of Bell inequalities are in principle possible, by using continuous-variable measurements and physically meaningful—though hardly feasible—quantum states.

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## APPENDIX: EXPRESSION IN THE FOCK BASIS

An interesting point is to study the decomposition of states $|f\rangle$ and $|g\rangle$ on the Fock basis $|n\rangle$, which are experimentally accessible. Starting from states (32), (33) with $\alpha = \sqrt{\pi}$, $s=0.4$, and with $N$ satisfying condition (36), we obtained after truncating at the 14th order and normalizing the states:

$$|f\rangle = \sqrt{0.459}|0\rangle - \sqrt{0.491}|4\rangle - \sqrt{0.008}|8\rangle - \sqrt{0.042}|12\rangle, \tag{A1}$$

$$|g\rangle = \sqrt{0.729}|1\rangle + \sqrt{0.155}|5\rangle - \sqrt{0.107}|9\rangle - \sqrt{0.009}|13\rangle. \tag{A2}$$

These states allow to reach a violation $S-2=0.81$. The state $|f\rangle$ only involves orders $n \equiv 0 \pmod 4$, as we have $n \equiv 1 \pmod 4$ for $|g\rangle$. This directly comes from the fact that $\langle p|n\rangle = (-i)^n \langle q|n\rangle_{q=p}$ and $(-i)^4 = 1$, so that states of the form $\Sigma_{n\equiv a\ (\mathrm{mod}4)} c_n|n\rangle$ are eigenvectors of the Fourier transform. From these considerations we have obtained $S=2.68$ for

$$|f\rangle = \sqrt{0.585}|0\rangle - \sqrt{0.415}|4\rangle, \tag{A3}$$

$$|g\rangle = \sqrt{0.848}|1\rangle + \sqrt{0.152}|5\rangle. \tag{A4}$$

and $S=2.3$ for

$$|f\rangle = \sqrt{0.67}|0\rangle - \sqrt{0.33}|4\rangle, \quad |g\rangle = |1\rangle. \tag{A5}$$

These states are quite simple, and it is expected that a specific simplified procedure to produce them might be designed.

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