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Engineering three-body interaction and Pfaffian states in circuit QED systems

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We demonstrate a scheme to engineer the three-body interaction in circuit-QED systems by tuning a fluxonium qubit. Connecting such qubits in a square lattice and controlling the tunneling dynamics in the form of a synthesized magnetic field for the photon-like excitations of the system allows the implementation of a parent Hamiltonian whose ground state is the Pfaffian wave function. Furthermore, we show that the addition of the next-nearest-neighbor tunneling stabilizes the ground state, recovering the expected topological degeneracy even for small lattices. Finally, we discuss the implementation of these ideas with the current technology.

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Many-body topological states have fascinating properties such as non-Abelian statistics that have been theoretically predicted but have not been observed [1]. Such states have also been proposed as a promising platform to perform robust quantum computation [2]. The simplest state with non-Abelian properties was constructed by Moore and Read in the form of a Pfaffian wave function, in the context of fractional quantum Hall effect [3]. At the same time, "parent Hamiltonians" have been introduced to generate such states as their ground states. In particular, a Hamiltonian with three-body interaction was proposed by Greiter *et al.* which yields the Pfaffian state [4]. There have been remarkable efforts to generate such Hamiltonians, e.g., using an ultracold-atom system [5,6]; however, the elimination of the two-body interaction while preserving the bosonic nature of excitations remains challenging [7–11], as expected for perturbatively generated three-body terms [12]. In this Rapid Communication, we present a scheme using circuit QED systems with ultrastrong microwave nonlinearity [13,14] to achieve this end. We demonstrate how to engineer a three-body interaction and a synthetic magnetic field which are both required to implement the parent Hamiltonian of Ref. [4] in a lattice.

The key idea in this Rapid Communication is to introduce a generalized qubit that exhibits the three-body interaction; i.e., one and two excitations are allowed in the qubit but the creation of the third excitation has an energy penalty, as shown in Fig. 1(b), compared to Fig. 1(a) for a two-body interaction. Such a qubit can be characterized by a Hamiltonian of the form $H_i = \frac{1}{6}U_3\hat{a}_i^{\dagger 3}\hat{a}_i{}^3$, where $\hat{a}_i$ is the bosonic creation operator at site i and U_3 is the interaction strength. This qubit can be generated by tuning various parameters of a fluxonium qubit [15], in a parameter regime similar to that of a transmon [16], to achieve the desired level structure. We couple these qubits in a square lattice, using SQUIDs where their external phase is modulated, as shown in Fig. 1(c). This modulation can imprint a tunneling phase [17], which is arranged to implement a synthetic magnetic field with a fixed gauge. Such a system can be described by the following Hamiltonian:

$$H = -J\sum_{x,y}\hat{a}_{x+1,y}^{\dagger}\hat{a}_{x,y}e^{-i2\pi\alpha y} + \hat{a}_{x,y}^{\dagger}\hat{a}_{x+1,y}e^{+i2\pi\alpha y} + \hat{a}_{x,y+1}^{\dagger}\hat{a}_{x,y} + \hat{a}_{x,y}^{\dagger}\hat{a}_{x,y+1} + \frac{1}{6}U_3\hat{a}_{x,y}^{\dagger 3}\hat{a}_{x,y}^{3}, \tag{1}$$

where α is the acquired phase from tunneling around a single plaquette and characterizes the strength of the magnetic field. In the continuum limit ($\alpha \ll 1$), this Hamiltonian is known to be the parent Hamiltonian of Pfaffian states, when the filling factor is $\nu = N/N_\alpha = 1$, where N is the number of particles inside the system and N_α is the total number of magnetic flux. In other words, the Pfaffian state is the ground state and the excited states have non-Abelian statistics. We present our numerical results that indicate that indeed the ground state is the Pfaffian state with the threefold topological degeneracy. We show that adding next-nearest-neighbor tunneling can flatten the single-particle energy bands, and therefore, the Pfaffian state can be seen even for high magnetic fields ($\alpha = 0.25$).

In order to implement the Hamiltonian of Eq. (1), we need two key elements: (1) inducing a magnetic-type hopping between sites and (2) generating the three-body nonlinearity. We start by describing a single qubit that exhibits the three-body interaction and return to the discussion of the magnetic hopping later. We consider a fluxonium [15], which can be described by the following Hamiltonian:

$$H_{\text{qubit}} = 4E_c n^2 + \tfrac{1}{2}E_L\phi^2 - E_J\cos(\phi+\phi_x), \tag{2}$$

where E_c is the single-electron charging energy, E_J is the Josephson junction energy, and $E_L = (\Phi_0/2\pi)^2/L$ characterizes the shunted inductive energy, defined in terms of the flux quantum $\Phi_0 = h/2e$. (n,ϕ) are conjugate variables and are equal to $\sqrt[4]{8E_c/E_L}$(Cooper pair number, node flux), respectively. ϕ_x is the external flux through the Josephson junction in units of the magnetic flux quanta. The spectrum of this Hamiltonian, which basically describes a particle in an anharmonic potential, can be numerically obtained. We consider the so-called transmon regime where $E_J \gg E_c$ [16], so that the qubit could remain less sensitive to charge noise. We are interested in the limit where the first and the second excitation levels have the same energy and the third excitation is detuned from them [Fig. 1(b)]. The nonlinearity is provided

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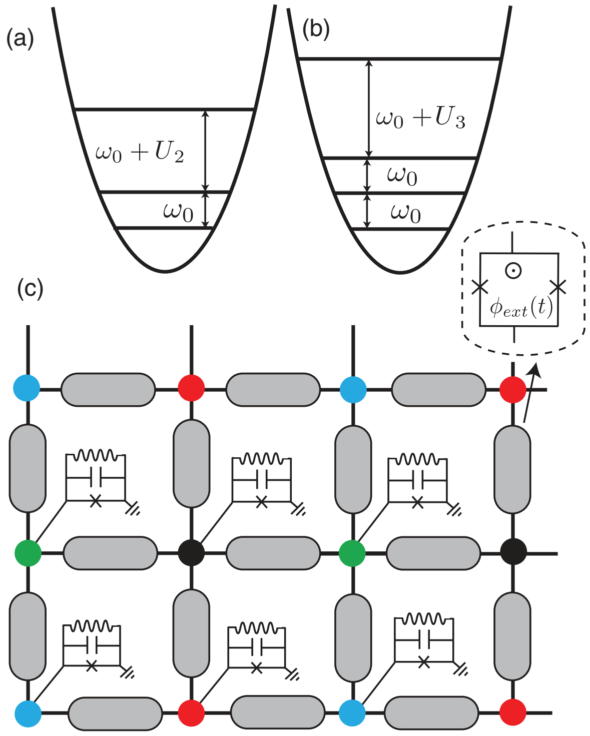

FIG. 1. (Color) (a) Two-body and (b) three-body interactions emerge as two- and three-excitation nonlinearity in an anharmonic oscillator. (c) Circuit model to implement Eq. (1). Each site is composed of an anharmonic oscillator with the three-body interaction. The qubits are detuned from each other according to a staggered pattern (blue/red/green/black) and are coupled by externally modulated SQUIDs.

by the Josephson junction and tuned to the desired form using the shunted inductor and the external flux. In particular, we analyze the four lowest energy eigenstates. In such subspace, one can describe the system with a general Hamiltonian of the form

$$H_{\text{model}} = \omega_0 \hat{a}^\dagger \hat{a} + \tfrac{1}{2} U_2 \hat{a}^{\dagger 2} \hat{a}^2 + \tfrac{1}{6} U_3 \hat{a}^{\dagger 3} \hat{a}^3, \tag{3}$$

where $\hat{a}^\dagger$ is the creation operator of a single excitation, ω_0 is the energy of the lowest level, and U_2 (U_3) characterizes the two- (three-) particle interaction, as shown in Figs. 1(a) and 1(b).

Figures 2(a) and 2(b) shows the numerical results for (U_2,U_3), respectively. We observe that for a given E_L, the external flux ϕ_x can be tuned so that the two-body nonlinearity vanishes. However, the three-body nonlinearity does not necessarily vanish for that specific ϕ_x. Figure 2(c) shows the value of the three-body nonlinearity when we operate at $U_2 = 0$. The largest value of U_3, while keeping the bosonic nature of the excitations (see the next paragraph), is achieved for the following parameters: $E_L \simeq 1.4 E_J$, $\phi_x \simeq 2.68$, for $E_c = 0.05 E_J$. This suggests that once such qubits are coupled to each other, the excitations can hop in between them, and only zero, one, and two Fock states on each site can be occupied. However, it is not guaranteed that the hopping has the correct bosonic form.

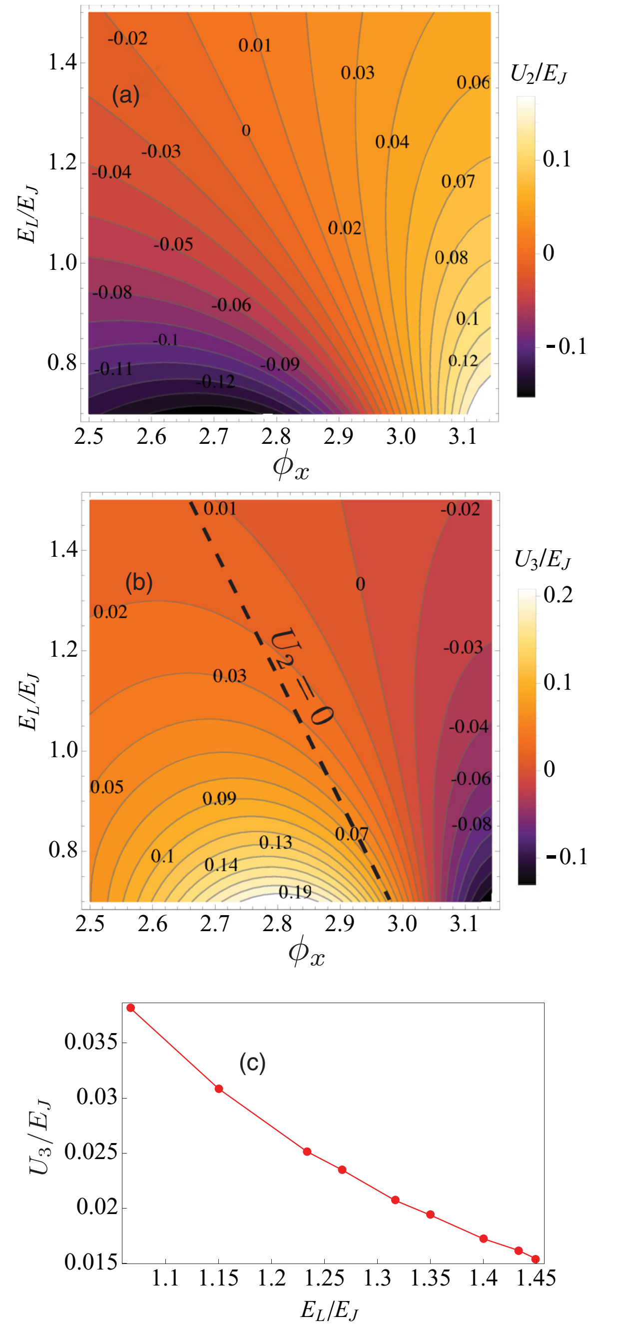

FIG. 2. (Color online) (a) Three-body (U_3) and (b) two-body (U_2) interaction strength as function of the external bias flux and the shunted inductance. The dashed line on (b) shows where the two-body interaction vanishes. (c) Shows the three-body interaction strength when U_2 is optimized to be less than $0.0005 E_J$. All the plots are for $E_c/E_J = 0.05$.

In order to verify that the hopping has the correct bosonic enhancement factor (for example $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$; see Supplemental Material [18]), we consider two qubits that are inductively coupled to each other [Fig. 3(a)] and analyze the dynamics of several excitations in between them. In particular, we consider the coupling of the form $H_{\text{coup}} = M\phi_1\phi_2$, where M is the tunneling energy, proportional to the mutual inductance, and ϕ_i is the phase of the qubit i.

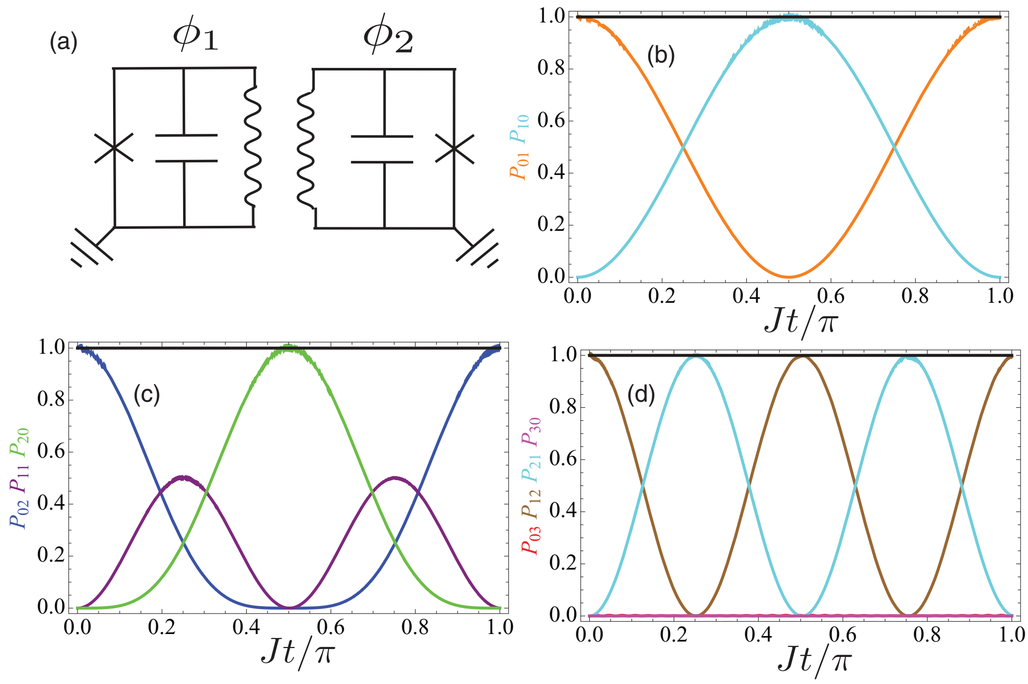

FIG. 3. (Color) Dynamics of different excitation numbers when two qubits are inductively coupled to each other, as shown in (a): (b) one excitation, (c) two excitations, (d) three excitations. The black line shows the total population in (b) and (c) and $P_{21} + P_{12}$ in (d).

First, we study the dynamics of a single excitation in the coupled system. We prepare one qubit in a Fock state with one excitation and let the system evolve. The population dynamics is plotted in Fig. 3(b). We observe that the system undergoes Rabi oscillations between two states. Next, we consider two excitations in the system. Similarly, we initiate the system with the two-excitation Fock state and study the dynamics. We see that the microscopic Hamiltonian of our system leads to a dynamic identical to that of a model system of two bosonic oscillators with two excitations, as shown in Fig. 3(c). Finally, we start with three excitations in the system, one in the first qubit and two in the second qubit, as shown in Fig. 3(d). Due to the presence of the three-body nonlinearity ($U_3 \neq 0$), we observe that the population in the three-excitation Fock states is suppressed (less than 10^{-7}), similar to a model system of two bosonic systems in the three-body hard-core limit ($\hat{a}_i^3|\Psi\rangle = 0$, where $|\Psi\rangle$ is an arbitrary state of the system).

Now, we discuss the implementation of the magnetic hopping terms in Eq. (1). There have been several proposals in the past to engineer a magnetic-like Hamiltonian in the context of circuit-QED systems [19,20] and also proposals without breaking the time reversal symmetry in photonic systems [21,22]. Here, we present a scheme based on phase modulating the SQUIDs that couple adjacent sites. In particular, we set the resonance (ω_i) of adjacent sites to be different from each other and form a staggered pattern, as shown in Fig. 1(c). The connecting SQUID inductance is modulated by applying a microwave flux $\phi_{\text{ext}}(t) = \delta\phi\cos(\Delta_{ij}t + \phi_p)$, in units of the magnetic flux quantum, where $\delta\phi \ll 1$ and the microwave pump frequency is tuned to the frequency difference of two adjacent sites ($\Delta_{ij} = \omega_i - \omega_j$). As shown in Ref. [17], such modulation induces a hopping Hamiltonian between two modes of the form $a_i^\dagger a_j e^{+i\phi_p} + a_j^\dagger a_i e^{-i\phi_p}$, in the rotating frame with the rotating wave approximation. The difference between our case and Ref. [17] is that there the hopping was induced between two modes of the same waveguide, while here the hopping is induced between two modes of different sites. In contrast to the previous scheme proposed by Koch *et al.* [19], our scheme is not sensitive to charge noise, and the generated magnetic field is insensitive to minor device variations.

The Pfaffian state is the ground state of the Hamiltonian of Eq. (1) in the continuum limit ($\alpha \ll 1$). However, the lattice could distort the wave function, close the gap, and destroy the topological order. To map the lattice to the continuum in such models, the concept of long-range tunneling was suggested by Laughlin [23]. By introducing the long-range hopping, the single-particle spectrum becomes flat and the many-body gap is enhanced. As an example, for the Laughlin fraction ($\nu = 1/2$) in bosonic systems, for large magnetic field ($\alpha \gtrsim .4$) [24], the gap closes and the topological order of the ground state disappears. However, Kapit and Mueller showed that including long-range order tunneling flattens the lowest branch of the single-particle Hofstadter's spectrum and improves the gap even for large magnetic fields $\alpha \simeq 0.5$ [25]. Furthermore, the long-range tunneling allows braiding operation even for small lattices [26]. Here, we consider the Pfaffian fraction for bosons ($\nu = 1$) and we observe similar behavior. We assume the bosonic occupation number on each site cannot exceed 2; i.e., $U_3 = \infty$. In this situation, the ground state of the system on the torus should be threefold degenerate and should be separated by a gap [4]. As shown in Fig. 4(a), and previously reported in Ref. [10], the gap is nonzero even in a small 4×4 lattice. The ground state has the expected order; i.e., the Chern

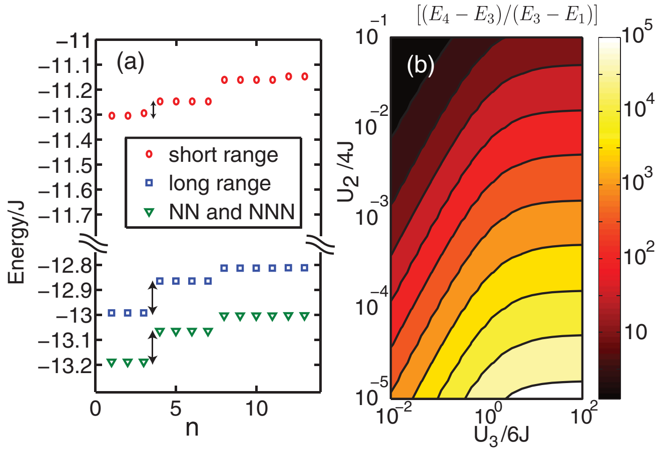

FIG. 4. (Color online) (a) First thirteen eigenvalues for 4 particles on a 4×4 lattice with torus boundary conditions, i.e., a total of 16 plaquettes, $\alpha = 0.25$, and $U_3 = \infty$. The index n labels eigenvalues. Different shapes show different tunneling schemes: circles for only nearest-neighbor tunneling (gap/$J \simeq 0.04$), triangles for the nearest- and the next-nearest-neighbor tunneling (gap/$J \simeq 0.12$), and squares for the long-range tunneling (gap/$J \simeq 0.12$). The gaps are denoted by arrows. (b) The order parameter as a function of U_3 and U_2, characterizing the relative size of the gap.

number is equal to 1 for the ground state manifold and a very weak overlap with the Pfaffian state, as expected for the lattice. Now, we consider the next-nearest-neighbor tunneling terms which in the Landau gauge are described as

$$\sum_{x,y}\sum_{\Delta x,\Delta y}(-1)^{\Delta x+\Delta y+\Delta x\Delta y}e^{-\frac{\pi}{2}(1-\alpha)(\Delta x^2+\Delta y^2)} \times e^{-i2\pi\alpha(y\Delta x+\Delta x\Delta y/2)}\hat{a}^\dagger_{x+\Delta x,y+\Delta y}\hat{a}_{x,y},$$

where Δx (Δy) is the number of site tunneling in the x (y) direction, respectively. If we consider only ($\Delta x = 1, \Delta y = 0$) and ($\Delta x = 0, \Delta y = 1$) terms, we recover the tunneling terms in Eq. (1). Since our implementation of magnetic field with modulation requires two connecting sites to have different frequencies, we choose a staggered patterned with four colors, as shown in Fig. 1(c), to implement the next-nearest-neighbor tunneling terms. As shown in Fig. 4(a), if we include next-nearest-neighbor terms, the threefold ground state degeneracy is preserved and the gap is improved threefold. In contrast to atoms on optical lattices, here in circuit-QED systems, the long-range tunneling term can be implemented by linking different sites using extra connecting SQUIDs.

In an experimental realization, one might not be able to access very large three-body interaction ($U_3 \gg J$), and entirely suppress the two-body interaction ($U_2 = 0$); therefore, it is important to assure that the gap between the threefold-degenerate ground state manifold and the excited state is preserved for a nonideal situation. To numerically investigate that, we define an "order parameter" as the ratio between the gap and the energy difference within the ground state; i.e., $\lambda = (E_4 - E_3)/(E_3 - E_1)$, where E_i is the energy of the ith eigenstate. As we see in Fig. 4(b), the system is completely gapped for large U_3 and zero U_2. As U_3 becomes small and U_2 becomes large, the order parameter vanishes, the gap closes. Therefore, we see that for a certain region in the parameter space the gap exists and the system is robust against the presence of a finite two-body interaction.

We next consider experimental issues involving the realization and detection of Pfaffian states in the proposed circuit QED system. As we discussed above the three-body interaction should be larger than the tunneling $U_3 \gg J$; i.e., we should choose the largest (smallest) possible U_3 (J), respectively. On the one hand, the three-body interaction is bounded from above as a small fraction of the Josephson energy. Considering a Josephson energy of tens of GHz, one can achieve a three-body interaction strength of a few 100 MHz. On the other hand, J is bounded from below, since the tunneling process should be faster than any decoherence mechanism. In particular, $J \gg T_2^{-1}, T_1^{-1}$, where T_1 (T_2) is the relaxation (decoherence) time, respectively. Recent experiments have shown $T_1, T_2 \gg 10\ \mu s$ [14]; therefore, assuming a tunneling rate of $J \geqslant (2\pi)10$ MHz guarantees many tunnelings to occurs before the coherence is lost. Therefore, one can achieve a regime where $U_3 \gtrsim 60J$, which according to Fig. 4(b) provides a sufficient energy gap between the ground manifold and the excited states.

As an initial experimental step to verify whether a three-body interaction has been implemented, we suggest a correlation function measurement. Specifically, when a single site with the Hamiltonian of Eq. (3) is driven with a weak coherent field, U_2 and U_3 can be obtained from measuring the output correlation functions $g^{(2)} = \langle \hat{a}^{\dagger 2}\hat{a}^2\rangle/\langle \hat{a}^\dagger \hat{a}\rangle^2$ and $g^{(3)} = \langle \hat{a}^{\dagger 3}\hat{a}^3\rangle/\langle \hat{a}^\dagger \hat{a}\rangle^3$, respectively (see Supplemental Material [18]). Such correlation function measurements have been successfully achieved in the microwave domain [27] using quadrature amplitude measurements instead of the conventional Hanbury Brown–Twiss measurements [28,29]. Alternatively, we can perform nonlinear spectroscopy to map out the anhormonic levels [30,31].

In summary, we have presented a scheme to implement the three-body interaction and the Hamiltonian to generate the Pfaffian state in circuit-QED systems. Due to relatively slow decoherence in these systems, one can use an adiabatic approach to prepare such states [32–34] and locally probe each site using an auxiliary transmon [35]. However, a more relevant regime for such photonic systems is to externally drive them and investigate their many-body nonequilibrium behavior [36,37] and analyze their incompressibility [34,38–40]. Another intriguing direction is to explore the braiding of non-Abelian anyons in such systems. It has been recently shown that even in small systems with long-range tunneling such braiding can be obtained [26]. One can possibly dynamically detune the resonance frequency of the site to trap and move around the anyons.

We recently became aware of a similar proposal, concurrently developed by Kapit and Simon [41], using spin-1/2 systems.

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